Thursday, May 12, 2016 2:42 PM

$$\sum_{p\in A_{N,a}} \frac{1}{p^s} + O(1) = \sum_{p,m} \frac{\mathbb{1}_{N,a}(p^m)}{m\, p^{ms}}$$

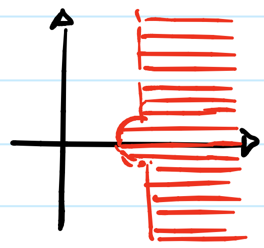
$$= \frac{1}{\varphi(N)} \sum_{\chi\in (\mathbb{Z}/N\mathbb{Z})^\times} \overline{\chi(a)} \sum_{p,m} \frac{1}{m}\left(\frac{\chi(p)}{p^s}\right)^m$$

$$= \frac{1}{\varphi(N)}\left[\sum_{p,m} \frac{1}{m}\left(\frac{\chi_0(p)}{p^s}\right)^m + \sum_{\chi\neq\chi_0} \overline{\chi(a)} \left(\sum_p -\operatorname{Ln}\left(1-\frac{\chi(p)}{p^s}\right)\right)\right]$$

- $e^{\sum_p -\operatorname{Ln}\left(1-\frac{\chi(p)}{p^s}\right)} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} =: L(\chi,s)$

- $L(\chi,s)$ is holomorphic on $\operatorname{Re}(s)>0$.

- Suppose $L(\chi,1)\neq 0$.



$\Rightarrow$ $\exists$ a holomorphic function $\ell(\chi,s)$ on
s.t. $e^{\ell(\chi,s)} = L(\chi,s)$

In particular, $\ell(\chi,s) = \sum_p -\operatorname{Ln}\left(1-\frac{\chi(p)}{p^s}\right) + 2\pi i\, n_\chi$.

So $\sum_{p\in A_{N,a}} \frac{1}{p^s} + O(1) = \frac{1}{\varphi(N)} \sum_p \frac{1}{p^s} + \frac{1}{\varphi(N)} \sum_{\chi\neq\chi_0} \overline{\chi(a)}\, \ell(\chi,s)$.

By Euler we have

$$\left.\begin{array}{l} \sum_p \frac{1}{p^s} + O(1) = \zeta(s) \Rightarrow \lim_{s\to 1^+} \sum_p \frac{1}{p^s} = \infty \\ \lim_{s\to 1^+} \ell(\chi,s) = \ell(\chi,1) < \infty \end{array}\right\} \Rightarrow$$

$$\lim_{s\to 1^+} \sum_{p\in A_{N,a}} \frac{1}{p^s} = \infty.$$

- To show the non-vanishing:

$$\zeta_N(s) := \prod_\chi L(\chi,s)$$

$$\Big| \qquad \qquad (\quad \chi(\;)^{-1}$$

For $\mathrm{Re}(s) > 1$, $\zeta_N(s) = \prod_p \prod_\chi \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$

$$= \prod_{p \nmid N} \left(1 - \frac{1}{p^{o(p)s}}\right)^{-\varphi(N)/o(p)}$$

$$\Rightarrow \zeta_N(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{for some } a_n \in \mathbb{Z}^{\geq 0} \text{ and } \mathrm{Re}(s) > 1.$$

- Then we showed that if $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has a holomorphic extension to $\mathrm{Re}(s) > \alpha$ and $a_n \geq 0$, then $\sum_{n=1}^{\infty} \frac{a_n}{n^\alpha} < \infty$ (Landau Theorem)

- Observed that $\zeta_N(1/\varphi(N)) = \infty$ to get a contradiction.