Let's go back to Euler's method and how we got the initial information about $\sum_{p\le x}\frac{1}{p}$. We considered (here we can work with $s\in\mathbb{R}^{>1}$.)

$$\sum \frac{1}{n^s} = \prod_p \frac{1}{(1-1/p^s)}$$

$$\Rightarrow \ln\left(\sum \frac{1}{n^s}\right) = \sum_p -\ln\left(1-\frac{1}{p^s}\right)$$

$$= \sum_{p,\, m\ge 1} \frac{1}{m\, p^{ms}}$$

$$= \sum_p \frac{1}{p^s} + O(1)$$

$\Rightarrow$ Since the LHS $\to\infty$ as $s\to 1^+$,

the RHS $\to\infty$ as $s\to 1^+$

$$\Rightarrow \sum_p \frac{1}{p^s} \longrightarrow \infty \quad \text{as } s\to\infty.$$

Dirichlet's idea was to reverse engineer this approach for

$$\sum_{p\in P_{N,a}} \frac{1}{p^s}$$

where $P_{N,a} := \{Nk+a \mid k\in\mathbb{Z}\}$.

The first step is O.K.:

$$\text{I} \qquad \sum_{p\in P_{N,a}} \frac{1}{p^s} + O(1) = \sum_{\substack{p\in P_{N,a}\\ m\ge 1}} \frac{1}{m\, p^{ms}}$$

This, however, is not $\underline{\ln}$ of a Euler-product-type of formula. Let's recall that in one of your homework assignments you have proved:

If $f:\mathbb{Z}^+\to\mathbb{C}$ is a strictly multiplicative and $|f(n)|\le 1$, then $$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\prod_p \frac{1}{1-\frac{f(p)}{p^s}} \quad \text{for } \operatorname{Re}(s)>1$$

(and both sides are convergent.)

So in order to have a Euler-product-type of formula, one needs to have (strictly) multiplicative function. Let's examine

I: what kind of function is involved?

$$\sum_{p\in P_{N,a}}\frac{1}{p^s}=\sum_p\frac{\mathbb{1}_{N,a}(p)}{p^s} \quad \text{where } \mathbb{1}_{N,a}(m)=\begin{cases}1 & m\in P_{N,a}\\ 0 & m\notin P_{N,a}.\end{cases}$$

And it seems we need to change I to

I′ $$\sum_{p\in P_{N,a}}\frac{1}{p^s}+O(1)=\sum_{p,m\geq 1}\frac{\mathbb{1}_{N,a}(p^m)}{m\,p^{ms}},$$

but we are still stucked as $\mathbb{1}_{N,a}$ is NOT necessarily a (strictly) multiplicative function.

Notice that $\mathbb{1}_{N,a}$ factors through $\mathbb{Z}/N\mathbb{Z}$, i.e.

$$n_1\equiv n_2 \pmod N \Rightarrow \mathbb{1}_{N,a}(n_1)=\mathbb{1}_{N,a}(n_2).$$

So it is essentially a function $\mathbb{Z}/N\mathbb{Z}\to\mathbb{C}$.

Since $\gcd(a,N)=1$ and we are "looking for" multiplicative functions it is better to think about $\mathbb{1}_{N,a}$ as a function

$$(\mathbb{Z}/N\mathbb{Z})^{\times}\longrightarrow\mathbb{C}.$$

Dirichlet is a German mathematician and, when he spent time in Paris, he was a friend of Fourier. So he learned Fourier analysis from him. In the context that we need it implies the following:

<u>Theorem</u> For any finite abelian group $G$,

$$\hat{G}:=\operatorname{Hom}(G,S^1)$$

forms an orthonormal basis of the vector space $\{f : G \to \mathbb{C}\}$

where $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\, \overline{f_2(g)}$.

(We will prove this later.)

Corollary. Suppose $G$ is a finite abelian group. Then

$$\forall g_0 \in G, \qquad \mathbb{1}_{g_0} = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \overline{\chi(g_0)}\, \chi,$$

where $\mathbb{1}_{g_0}(g) = \begin{cases} 1 & \text{if } g = g_0, \\ 0 & \text{otherwise.} \end{cases}$

Proof. Since $\widehat{G}$ is an orthonormal basis,

$$\mathbb{1}_{g_0} = \sum_{\chi \in \widehat{G}} \langle \mathbb{1}_{g_0}, \chi \rangle \chi. \text{ And}$$

$$\langle \mathbb{1}_{g_0}, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \mathbb{1}_{g_0}(g)\, \overline{\chi(g)} = \frac{\overline{\chi(g_0)}}{|G|}. \quad \blacksquare$$

So by the above Corollary

$$\forall m \in (\mathbb{Z}/N\mathbb{Z})^\times, \quad \mathbb{1}_{N,a}(m) = \frac{1}{\varphi(N)} \sum_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} \overline{\chi(a)}\, \chi(m).$$

Any $\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}$ gives us a function from $\mathbb{Z}^+$ to $S^1$ (that we again denote by $\chi$):

$$\chi(m) := \begin{cases} \chi(m + N\mathbb{Z}) & \text{if } \gcd(m, N) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that these are strictly multiplicative.

These are called Dirichlet characters.

Altogether we get

$$\sum_{p \in P_{N,a}} \frac{1}{p^s} + O(1) = \sum_{p,\, m \geq 1} \frac{\mathbb{1}_{N,a}(p^m)}{m\, p^{ms}}$$

$$= \sum \frac{1}{\varphi(N)} \sum_{\chi \in \widehat{(\mathbb{Z}/ \quad)^\times}} \overline{\chi(a)}\, \chi(p^m)$$

II

$$= \sum_{p,\, m \geq 1} \frac{\frac{1}{\varphi(N)} \sum_{\chi \in (\mathbb{Z}/N\mathbb{Z})^\times} \overline{\chi(a)}\, \chi(p^m)}{m\, p^{ms}}$$

$$= \frac{1}{\varphi(N)} \sum_{\chi \in (\mathbb{Z}/N\mathbb{Z})^\times} \overline{\chi(a)} \underbrace{\sum_{p,\, m \geq 1} \frac{\chi(p)^m}{m\, p^{ms}}}_{\ell(\chi, s)}$$

$$= \frac{1}{\varphi(N)} \ell(\chi_0, s) + \frac{1}{\varphi(N)} \sum_{\chi \neq \chi_0} \ell(\chi, s)$$

where $\chi_0$ is the trivial character, i.e.

$$\chi_0(m) = \begin{cases} 1 & \text{if } \gcd(m, N) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\ell(\chi_0, s) = \sum_{\substack{p,\, m \geq 1 \\ p \nmid N}} \frac{1}{m\, p^{ms}} = \sum_p \frac{1}{p^s} + O(1)$

$\boxed{\text{Only finitely many primes divide } N}$

So $\ell(\chi_0, s) \xrightarrow[s \to 1^+]{} \infty$. Hence to get the desired result it is enough to show

$\ell(\chi, s)$ stays bounded as $s \to 1^+$ if $\chi \neq \chi_0$.

For $\mathrm{Re}(s) > 1$, $\sum_{p,\, m \geq 1} \frac{\chi(p)^m}{m\, p^{ms}}$ is absolutely convergent

$$\Rightarrow \ell(\chi, s) = \sum_p \left( \sum_{m \geq 1} \frac{1}{m} \left( \frac{\chi(p)}{p^s} \right)^m \right)$$

Since $\left| \frac{\chi(p)}{p^s} \right| = \frac{1}{p^{\mathrm{Re}(s)}} < 1$, we have

$$-\mathrm{Ln}\left(1 - \frac{\chi(p)}{p^s}\right) = \sum_{m \geq 1} \frac{1}{m} \left( \frac{\chi(p)}{p^s} \right)^m \quad \text{for } \mathrm{Re}(s) > 1.$$

$$\Rightarrow \ell(\chi, s) = \sum_p -\mathrm{Ln}\left(1 - \frac{\chi(p)}{p^s}\right).$$

$\ell(\chi, s) \qquad \sum_p -\mathrm{Ln}\left(1 - \frac{\chi(p)}{p^s}\right)$

$$\Rightarrow \quad e^{\ell(\chi,s)} = e^{\sum_p -\mathrm{Ln}\left(1-\frac{\chi(p)}{p^s}\right)}$$

we need to show this

$$= \prod_p e^{-\mathrm{Ln}\left(1-\frac{\chi(p)}{p^s}\right)}$$

$$= \prod_p \frac{1}{1-\frac{\chi(p)}{p^s}}$$

as $\chi$ is strictly multip. and $|\chi| \le 1$

$$= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} =: L(\chi,s)$$

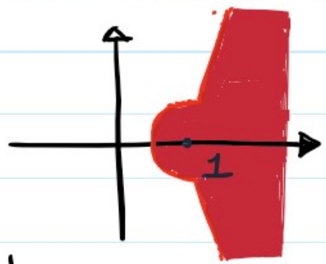
These are called Dirichlet L-functions. So to show $\ell(\chi,s)$ stays bounded as $s \to 1^+$, one should at least show

⊛ $\lim_{s\to 1^+} L(\chi,s)$ is not $0$ and can be extended holomorphically to $\mathrm{Re}(s) > 1-\varepsilon$.

(In fact, it will be extended to $\mathrm{Re}(s) > 0$.)

Before we prove ⊛, let's see why ⊛ is enough in order to get that $\ell(\chi,s)$ stays bounded as $s \to 1^+$.

Since $L(\chi,s)$ is holomorphic on $\mathrm{Re}(s) > 1-\varepsilon$ and half-plane is open and simply-connected and $L(\chi,s) \neq 0$ for $s$ in $U$,




there is a holomorphic function $g: U \to \mathbb{C}$ s.t. $L(\chi,s) = e^{g(s)}$.

Hence $g(s) = \ell(\chi,s) + 2\pi i\, n(s)$ for $\mathrm{Re}(s) > 1$ and $s \in U$

$\Rightarrow$ $n(s)$ is holomorphic on $\mathrm{Re}(s) > 1$ and $s \in U$
$n(s) \in \mathbb{Z}$

$\Rightarrow$ $n(s) = n_0$ is constant.

$\Rightarrow \quad \ell(\chi,s) = g(s) - 2\pi i\, n_0$ for $\mathrm{Re}(s) > 1$ and $s \in U$

$\ell$ h h implies $\ell(\chi,s)$

$$\Rightarrow \lim_{s\to 1^+} L(\chi,s) = g(1) - 2\pi i\, r_0 \text{, which implies } L(\chi,s)$$

stays bounded as $s \to 1^+$.

<u>Proposition</u>. If $\chi \neq \chi_0$, then $\displaystyle\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ is convergent for $\mathrm{Re}(s)>0$.

And it is uniformly on compact subsets of $\mathrm{Re}(s)>0$. Hence

$L(\chi,s)$ is holomorphic on $\mathrm{Re}(s)>0$.

<u>Proof.</u> Let $C(x) = \displaystyle\sum_{n\leq x} \chi(n)$. Then

$$\sum_{n\leq m} \frac{\chi(n)}{n^s} = \frac{C(m)}{m^s} + s\int_1^m \frac{C(t)}{t^{s+1}}\,dt$$

$$\left\{ \begin{array}{l} \displaystyle\sum_{Nk\leq n<Nk+N} \chi(n) = 0 \quad \text{as } \langle \chi, \chi_0\rangle = 0 . \\ \Rightarrow \quad C(x) \leq N . \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} \displaystyle\int_1^{\infty} \frac{C(t)}{t^{s+1}}\,dt \text{ is absolutely convergent for } \mathrm{Re}(s)>0 \\ \text{and } \displaystyle\frac{C(m)}{m^s} \xrightarrow[m\to\infty]{} 0 \quad \text{for } \mathrm{Re}(s)>0 . \end{array} \right.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \text{ is convergent.}$$

(Ex. show why it is uniform on bounded closed subsets.) ∎

So it is enough to show

$$L(\chi,1) \neq 0 \quad \text{if} \quad \chi \neq \chi_0 .$$

Let's consider $\displaystyle\prod_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} L(\chi,s) =: \zeta_N(s)$.

For $\mathrm{Re}(s)>1$, we have

$$\zeta_N(s) = \prod_{\chi\in\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} \prod_p \frac{1}{1-\frac{\chi(p)}{p^s}}$$

$$= \prod \left( \prod \left(1 - \frac{\chi(p)}{\quad}\right)\right)^{-1}$$

$$= \prod_p \left( \prod_\chi \left(1 - \frac{\chi(p)}{p^s}\right)\right)$$

- If $p \mid N$, then $\chi(p) = 0 \Rightarrow \prod_\chi \left(1 - \frac{\chi(p)}{p^s}\right) = 1$.

- If $p \nmid N$, then $p + N\mathbb{Z} \in (\mathbb{Z}/N\mathbb{Z})^\times$.

  For any finite abelian group $G$ and any $g \in G$ we have

  $$\prod_{\chi \in \hat{G}} (T - \chi(g)) = (T^{o(g)} - 1)^{|G|/o(g)}$$

  where $o(g)$ is the order of $g$ in $G$.

  (We will prove this later.)

Hence we have

$$(**) \qquad \zeta_N(s) = \prod_{p \nmid N} \left(1 - \frac{1}{p^{\mathrm{ord}_N(p)\, s}}\right)^{-\frac{\varphi(N)}{\mathrm{ord}_N(p)}}$$

for $\mathrm{Re}(s) > 1$, and if $L(\chi, 1) = 0$ for some $\chi$, then $\zeta_N(s)$ is holomorphic on $\mathrm{Re}(s) > 0$.

Let's focus on values of right hand side of the above equation at real values for $s$. For any $s \in \mathbb{R}^+$,

$$\left(1 - \frac{1}{p^{\mathrm{ord}_N(p)\, s}}\right)^{-\frac{\varphi(N)}{\mathrm{ord}_N p}} \geq \left(1 - \frac{1}{p^{\varphi(N)s}}\right)^{-1}.$$

$$\Rightarrow \prod_{p \nmid N} \left(1 - \frac{1}{p^{\mathrm{ord}_N(p)\, s}}\right)^{-\frac{\varphi(N)}{\mathrm{ord}_N p}} \geq \prod_{p \nmid N} \left(1 - \frac{1}{p^{\varphi(N)s}}\right)^{-1}$$

$$= \zeta(\varphi(N)s) \cdot \prod_{p \mid N} \left(1 - \frac{1}{p^{\varphi(N)s}}\right)$$

$\Rightarrow$ The right hand side of $(**)$ diverges at $s = \frac{1}{\varphi(N)}$.

It seems contradictory: the RHS has a holomorphic to $\mathrm{Re}(s) > 0$, but itself diverges at $s = \frac{1}{\varphi(N)}$. Can we get a contradiction?

So we focus on the RHS. Notice that $(1 + x + x^2 + \cdots)^m = \sum_{k=0}^{\infty} c_k x^k$ where $c_k \in \mathbb{Z}^+$. (We do not need this here, but it is worth mentioning where $c_k \in \mathbb{Z}^+$. (We do not need this here, but it is worth mentioning that there is a binomial expansion even for negative powers:

$$(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{-m}{k} (-1)^k x^k \quad \text{where}$$

$$\binom{-m}{k} = \frac{(-m)(-m-1)\cdots(-m-k+1)}{k!} = (-1)^k \binom{m-k+1}{k}$$

So $(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{m-k+1}{k} x^k$ for $|x|<1$. )

Hence $$\zeta_N(s) = \prod_{p \nmid N} \left(1 + c_{1,p} \frac{1}{p^{(\mathrm{ord}_N p)s}} + c_{2,p} \frac{1}{p^{2(\mathrm{ord}_N p)s}} + \cdots\right)$$

for some $c_{i,p} \in \mathbb{Z}^+$ and $\mathrm{Re}(s) > 1$.

$$\Rightarrow \zeta_N(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \quad \text{for some } \underline{b_n \in \mathbb{Z}^{\geq 0}} \text{ and } \underline{\mathrm{Re}(s) > 1}.$$

<u>Summary of we have got:</u> Some $b_n \in \mathbb{Z}^{\geq 0}$ such that

(1) $\sum_{n=1}^{\infty} \frac{b_n}{n^s}$ converges absolutely for $\mathrm{Re}(s) > 1$.

(2) $\sum_{n=1}^{\infty} \frac{b_n}{n^s}$ diverges for $s = 1/\varphi(N)$.

(3) $\sum_{n=1}^{\infty} \frac{b_n}{n^s}$ has a holomorphic extension on $\mathrm{Re}(s) > 0$.

The following theorem due to Landau gives us a contradiction.

<u>Theorem</u>. Suppose $a_n \in \mathbb{R}^{\geq 0}$. If $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is convergent on $\mathrm{Re}(s) > \rho$ and has a holomorphic extension on an open disk around $\rho$, then $\exists\, \varepsilon > 0$ s.t. $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges on $\mathrm{Re}(s) > \rho - \varepsilon$.

Before we prove the above theorem, let's see how it gives us a contradiction: Let $s_0 := \inf \left\{ s \in \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} \frac{b_n}{n^s} \text{ converges} \right\}$.

By (2), $\quad s_0 \geq 1/\varphi(N) > 0$.

By (2), $\quad s_0 \geq 1/\varphi(N) > 0$.

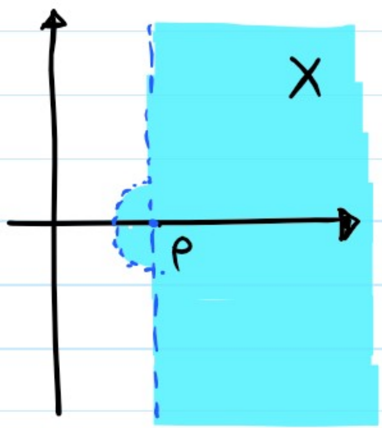
So $\sum_{n=1}^{\infty} \frac{b_n}{n^s}$ cannot have even a continuous extension to $\mathrm{Re}(s) > 0$, let alone holomorphic.

<u>Proof of Landau's theorem.</u> First observe that if for some $\alpha \in \mathbb{R}$ $\sum_{n=1}^{\infty} \frac{a_n}{n^\alpha} < \infty$, then $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is absolutely convergent, and uniformly on compact sets on $\mathrm{Re}(s) > \alpha$.

$\mathrm{Re}(s) > \alpha \Rightarrow \left|\frac{a_n}{n^s}\right| \leq \frac{a_n}{n^\alpha} \Rightarrow$ By comparison, $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is abs. conv.

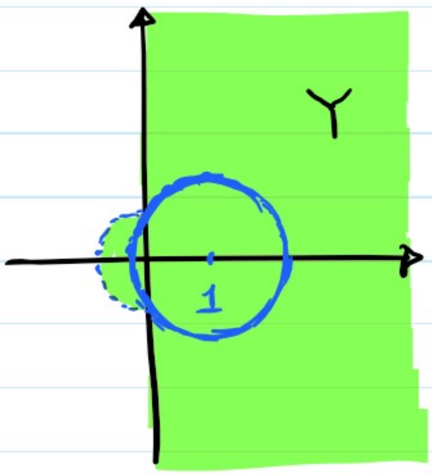
Using the fact that
$$\left|\frac{1}{n^{s_1}} - \frac{1}{n^{s_2}}\right| = \left|\int_{s_1}^{s_2} (-\ln n)\, n^{-z}\, dz\right|$$
$$\leq \ln(n) \int_{s_1}^{s_2} n^{-\mathrm{Re}(z)} |dz|$$
$$\leq \ln(n) \cdot n^{-\min\{\mathrm{Re}(s_1), \mathrm{Re}(s_2)\}} |s_1 - s_2|,$$
it is easy to see that, on $\mathrm{Re}(s) > \alpha$, $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges uniformly on compact sets. And so on $\mathrm{Re}(s) > \alpha$ it is holomorphic.

Let $f$ be the holomorphic function on $X$, and $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $\mathrm{Re}(s) > \rho$.




For simplicity, let's shift the whole thing to assume $\rho = 0$, I.e. let $g(s) = f(s - \rho)$.

So $g$ is holomorphic on $Y$ and
$$g(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \quad \text{where } c_n = \frac{a_n}{n^\rho} \geq 0.$$




For some $\varepsilon > 0$, the open disk centered at 1 with radius $1+2\varepsilon$ is a subset of $Y$. So $g(s)$ is equal to its Taylor expansion on this open disk; and Taylor series converges abs.

$$|s-1|<1+2\varepsilon \Rightarrow g(s)=\sum_{k=0}^{\infty}\frac{g^{(k)}(1)}{k!}(s-1)^k .$$

$$\frac{d}{ds}(n^{-s})=\frac{d}{ds}\left(e^{-(\ln n)s}\right)=-(\ln n)\,e^{-(\ln n)s}=-(\ln n)\,n^{-s}.$$

$$\Rightarrow \frac{d^k}{(ds)^k}(n^{-s})=(-\ln n)^k\,n^{-s}.$$

$$\Rightarrow g^{(k)}(s)=\sum_{n=1}^{\infty}\frac{(-\ln n)^k c_n}{n^s} \quad \text{on } \operatorname{Re}(s)>0.$$

$$\Rightarrow g^{(k)}(1)=\sum_{n=1}^{\infty}\frac{(-\ln n)^k c_n}{n}.$$

$$\Rightarrow g(s)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=1}^{\infty}\frac{(-\ln n)^k c_n}{n}\right)(s-1)^k \quad \text{if } |s-1|<1+2\varepsilon$$

$$\Rightarrow g(-\varepsilon)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=1}^{\infty}\frac{(\ln n)^k(1+\varepsilon)^k c_n}{n}\right)$$

$$=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left((1+\varepsilon)\ln n\right)^k\right)\frac{c_n}{n}$$

Since it is absolutely conv.

$$=\sum_{n=1}^{\infty}e^{(1+\varepsilon)\ln n}\cdot\frac{c_n}{n}=\sum_{n=1}^{\infty}\frac{c_n}{n^{-\varepsilon}}.$$

So by the first part $\sum_{n=1}^{\infty}\frac{c_n}{n^s}$ is absolutely convergent on $\operatorname{Re}(s)>-\varepsilon \Rightarrow \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ is abs. conv. on $\operatorname{Re}(s)>\rho-\varepsilon$. ∎