Necessary materials from complex analysis

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<u>Definition</u>. <u>Complex differentiation</u> of $f: U \to \mathbb{C}$ at z_0 is

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and it is denoted by $f'(z_0)$.

We say f is <u>holomorphic</u> on an open ball B if $\forall z \in B$, f is complex differentiable.

<u>Connection with real differentiation</u>.

$$\lim_{t \to 0} \frac{(u(z_0+t\omega) - u(z_0)) + i(v(z_0+t\omega) - v(z_0))}{t\omega} = f'(z_0)$$

for any $\omega \in \mathbb{C} \setminus \{0\}$.

For $\omega = 1$, we get $f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

For $\omega = i$, we get $f'(z_0) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

$$\Rightarrow \frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \text{ and } \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0).$$

<u>Corollary</u>. U: open and simply-connected (with no holes)

f holomorphic on U

$\gamma: [0,1] \to U$ a differentiable simple closed curve.

$$\Rightarrow \int_\gamma f(z)\,dz = 0.$$

<u>Proof</u>. $\int_\gamma (u+iv)(dx+idy) = \int_\gamma (u\,dx - v\,dy) + i(u\,dy + v\,dx)$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx$$

$$= \iint -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,dA + i\iint \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,dA$$

Green's theorem

$= 0$. ■

<u>Corollary</u>. U : simply connected

f : holomorphic on U

$\gamma_1, \gamma_2 : [0,1] \longrightarrow U$ two diff. curves

$\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$

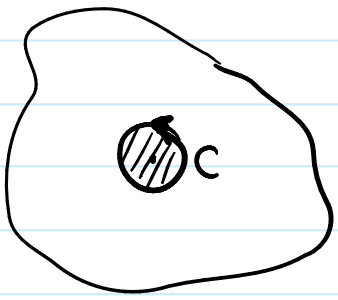
$$\Rightarrow \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz .$$

[So it just depends on the initial and the terminal points, and it is independent of path.].

<u>Cauchy integrals</u>.. f holomorphic on U.

. $D \subseteq U$ a closed disk centered at z_0

. C its boundary (circle).



$$\Rightarrow \begin{cases} f(z_0) = \dfrac{1}{2\pi i} \displaystyle\oint_C \dfrac{f(z)}{(z-z_0)}\,dz \\[2ex] f'(z_0) = \dfrac{1}{2\pi i} \displaystyle\oint_C \dfrac{f(z)}{(z-z_0)^2}\,dz . \end{cases}$$

<u>Proof</u>.

$$\frac{1}{2\pi i} \oint \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i} \oint \frac{f(z)-f(z_0)}{z-z_0}\,dz \quad \text{I}$$

$$+ \frac{1}{2\pi i} \oint \frac{f(z_0)}{z-z_0}\,dz . \quad \text{II}$$

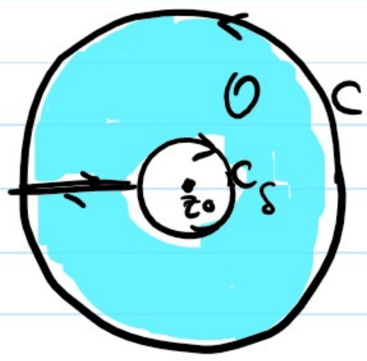
Let's start with II :

$$\frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0)}{\cancel{r_0} e^{i\theta}} \cancel{i}\, \cancel{r_0} e^{i\theta}\,d\theta$$

$$\boxed{\begin{aligned} dz &= dx + i\,dy \\ &= -r_0 \sin\theta\,d\theta + i\,r_0 \cos\theta\,d\theta \\ & i\,r\,e^{i\theta}\,d\theta \end{aligned}}$$

$$= i r_0 e^{i\theta}\, d\theta$$

$$= \frac{f(z_0)}{2\pi} 2\pi = f(z_0).$$



Now let's look at 1.

Since $\dfrac{f(z)-f(z_0)}{z-z_0}$ is holomorphic

on O, and its inter. is simply-connected,

we get $\displaystyle\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = \oint_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz$

where C_δ is the circle of radius δ centered at z_0.

$$\Rightarrow \left|\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| = \left|\oint_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right|$$

$$\leq \oint_{C_\delta} \left|\frac{f(z)-f(z_0)}{z-z_0}\right| |dz|$$

$$\leq \left(|f'(z_0)|+\varepsilon\right) \oint_{C_\delta} |dz|$$

$$\leq 2\pi\delta\left(|f'(z_0)|+\varepsilon\right) \longrightarrow 0$$

as $\delta \to 0$

$$\Rightarrow \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0.$$

So we have $\displaystyle f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz.$

$$f'(z_0) = \lim_{\omega\to 0} \frac{f(z_0+\omega)-f(z_0)}{\omega}$$

$$= \lim_{\omega\to 0} \frac{1}{\omega}\left(\oint_C \frac{f(z)}{z-(z_0+\omega)}\,dz - \oint_C \frac{f(z)}{z-z_0}\,dz\right)$$

$$= \lim_{\omega\to 0} \frac{1}{\omega} \oint_C f(z)\left(\frac{(z-z_0)-(z-z_0-\omega)}{(z-z_0-\omega)(z-z_0)}\right)dz$$

$$\omega \to 0 \qquad \omega \qquad C \qquad (z-z_0-\omega)(z-z_0)$$

$$= \lim_{\omega \to 0} \oint_C \frac{f(z)}{(z-z_0-\omega)(z-z_0)}\, dz$$

So it is enough to show

$$\lim_{\omega \to 0} \left| \oint_C \frac{f(z)}{(z-z_0-\omega)(z-z_0)} - \frac{f(z)}{(z-z_0)^2}\, dz \right| = 0.$$

$$\left| \frac{1}{(z-z_0-\omega)(z-z_0)} - \frac{1}{(z-z_0)^2} \right| = \left| \frac{1}{z-z_0} \right| \left| \frac{(z-z_0)-(z-z_0-\omega)}{(z-z_0)(z-z_0-\omega)} \right|$$

$$= \frac{|\omega|}{|z-z_0|^2 |z-z_0-\omega|}$$

And, $\forall z \in C$, $\quad |z-z_0| = r_0$ and $|z-z_0-\omega| \geq r_0 - \delta$ if $|\omega| < \delta$.

and $|f(z)| \leq M$ for some M.

So $\quad \left| \oint_C \frac{f(z)}{(z-z_0-\omega)(z-z_0)} - \frac{f(z)}{(z-z_0)^2}\, dz \right| \leq$

$$\frac{M}{r_0^2 (r_0 - \delta)}\, \delta\, (2\pi r_0) \xrightarrow[\delta \to 0]{} 0 \qquad \blacksquare$$

By a similar argument one can prove:

<u>Theorem</u>. If f is holomorphic on an open disk U, then it is n-step complex differentiable for any n and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz,$$

where C is a simple closed diff. curve around z_0 in U.

<u>Proposition</u>. f_n a sequence of holomorphic functions on U.

$f_n \to f$ converges uniformly on any closed disk.

$\Longrightarrow$ (1) f is holomorphic on U.

(2) $f_n' \to f'$ uniformly on any closed disk.

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<u>Proof.</u> $f_n(z_0) = \frac{1}{2\pi i} \oint_C \frac{f_n(z)}{z - z_0} dz$.

$\forall \varepsilon > 0$, if $n \gg_\varepsilon 1$, then $|f_n(z) - f(z)| \le \varepsilon \quad \forall z \in C$.

$$\Rightarrow \left| \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} - \frac{1}{2\pi i} \oint_C \frac{f_n(z)}{z - z_0} dz \right| \le$$

$$\frac{1}{2\pi} \oint_C \frac{|f(z) - f_n(z)|}{|z - z_0|} |dz| \le \frac{\varepsilon}{2\pi} \cdot \frac{2\pi r_0}{r_0} = \varepsilon.$$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{2\pi i} \oint_C \frac{f_n(z)}{z - z_0} dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

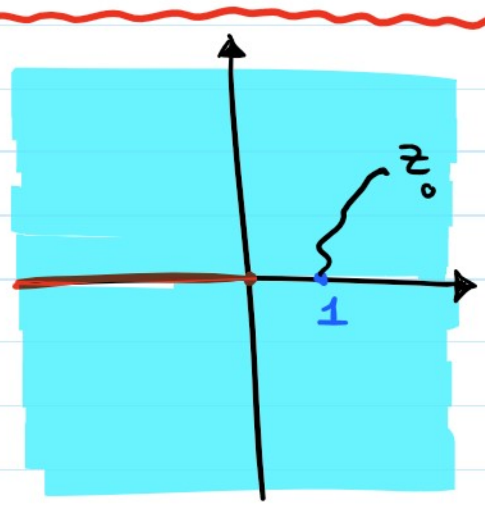
$$\lim_{n \to \infty} f_n(z_0) = f(z_0).$$

$\Rightarrow$ For any $z_0 \in U$, $f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz$.

$\Rightarrow$ f is holomorphic on U (?)

We can get the second part using $f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} dz$ and a similar argument as above. ∎

<u>Complex logarithm</u>



$\frac{1}{z}$ is holomorphic on U, and U is simply-connected. So for any curve $\gamma : [0,1] \to \mathbb{C} \setminus \mathbb{R}^{\le 0}$ (or any other ray which starts from the origin.) $\int_\gamma \frac{dz}{z}$ just depends on $\gamma(0)$ and $\gamma(1)$.

So we can define a function:

$$\mathrm{Ln} : \mathbb{C} \setminus \mathbb{R}^{\le 0} \to \mathbb{C}, \quad \mathrm{Ln}(z_0) := \int_\gamma \frac{dz}{z},$$

where $\gamma : [0,1] \to \mathbb{C} \setminus \mathbb{R}^{\leq 0}$, $\gamma(0) = 1$ and
$\gamma(1) = z_0$.

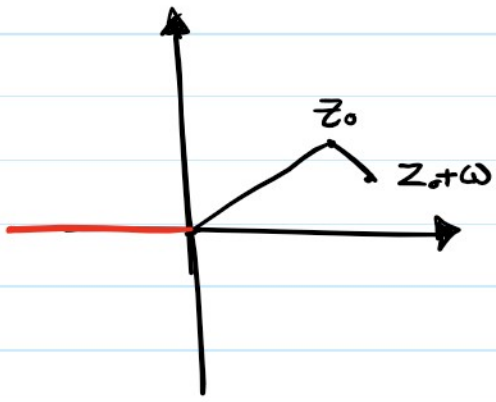
Basic Properties of Ln.

- $\forall\ x \in \mathbb{R}^+$, $\mathrm{Ln}(x) = \ln(x)$ (the usual natural logarithm).
- Ln is holomorphic on U and its derivative is $\frac{1}{z}$.

Pf. $\displaystyle \lim_{\omega \to 0} \frac{\mathrm{Ln}(z_0+\omega) - \mathrm{Ln}(z_0)}{\omega}$

$$= \lim_{\omega \to 0} \frac{1}{\omega} \int_0^1 \frac{1}{z_0 + t\omega}\, \omega\, dt$$

$$\boxed{\begin{aligned} dz = dx + i\,dy &\Leftarrow z = z_0 + t\omega \\ = \omega\, dt & \end{aligned}}$$



$$= \lim_{\omega \to 0} \int_0^1 \frac{dt}{z_0 + t\omega} = \frac{1}{z_0} + \lim_{\omega \to 0} \int_0^1 \left(\frac{1}{z_0 + t\omega} - \frac{1}{z_0} \right) dt$$

$$= \frac{1}{z_0} + \lim_{\omega \to 0} \int_0^1 \frac{t\omega}{(z_0 + t\omega)\, z_0}\, dt$$

$$\left| \int_0^1 \frac{t\omega}{(z_0+t\omega)\, z_0}\, dt \right| \leq \delta \frac{1}{|z_0|(|z_0| - \delta)} \xrightarrow[\delta \to 0]{} 0.$$

$\boxed{\text{if } |\omega| \leq \delta}$

$\blacksquare$

(if $z_1, z_2, z_1 z_2 \in \mathbb{R}^{\leq 0}$.)

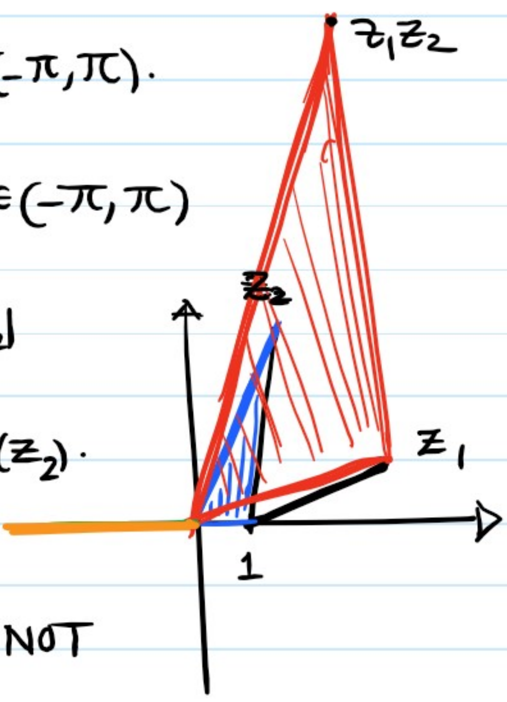
- $\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}(z_1) + \mathrm{Ln}(z_2) + 2\pi k i$ for some $k \in \{-1, 0, 1\}$.

Pf. $z_1 = r_1 e^{i\theta_1}$ for some $\theta_1 \in (-\pi, \pi)$.

$z_2 = r_2 e^{i\theta_2}$ for some $\theta_2 \in (-\pi, \pi)$

Multiplying by $z_2 \mapsto$ rescaling by $r_2 = |z_2|$

and rotating by $\theta_2 = \mathrm{Arg}(z_2)$.



Suppose the segment $\underline{z_1, z_1 z_2}$ does NOT intersect $\mathbb{R}^{\leq 0}$. Then

$z_1 z_2$

intersect $\mathbb{R}^{\leq 0}$. Then

$$\mathrm{Ln}(z_1 z_2) = \int_1^{z_1} \frac{dz}{z} + \int_{z_1}^{z_1 z_2} \frac{dz}{z}$$

$$= \mathrm{Ln}(z_1) + \int_1^{z_2} \frac{z_1\, d\omega}{z_1\, \omega}$$

$$= \mathrm{Ln}(z_1) + \mathrm{Ln}(z_2).$$

$$\boxed{\begin{array}{c} z = z_1 \omega \\ \Downarrow \\ dz = z_1\, d\omega \end{array}}$$

If the segment $\underline{z_1, z_1 z_2}$ intersects $\mathbb{R}^{\leq 0}$, then

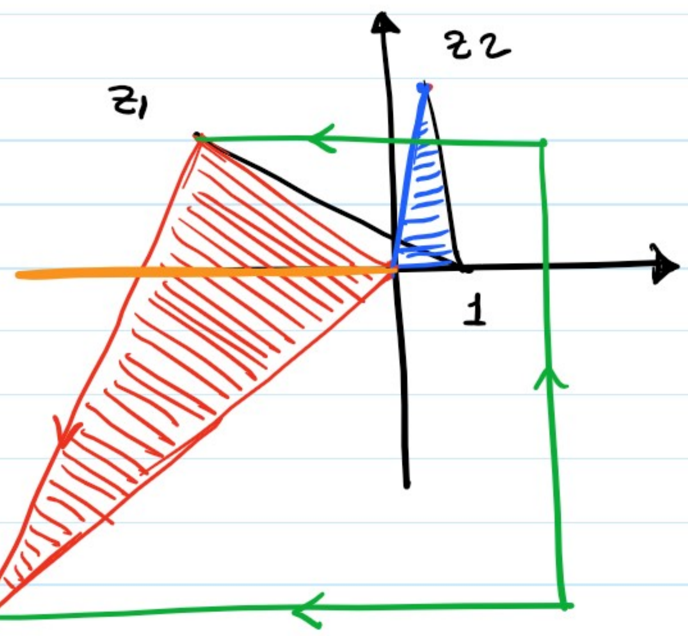
$$\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}(z_1) - \int_\gamma \frac{dz}{z}$$

$$= \mathrm{Ln}(z_1) + \int_{z_1}^{z_1 z_2} \frac{dz}{z}$$

$$- \oint_C \frac{dz}{z}$$

$$= \mathrm{Ln}(z_1) + \mathrm{Ln}(z_2)$$

$$\pm 2\pi i.$$

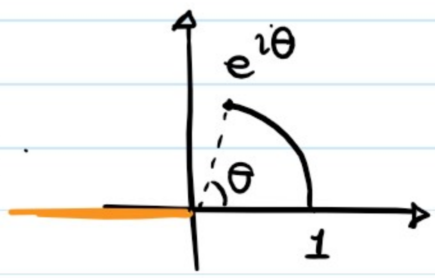


(sign depends on the orientation.)

- $\mathrm{Ln}(e^{i\theta_0}) = i\theta_0$ if $\theta_0 \in (-\pi, \pi)$.

$\underline{\text{Pf.}}$

$$\int_\gamma \frac{dz}{z} = \int_0^{\theta_0} \frac{i e^{i\theta}}{e^{i\theta}}\, d\theta = i\theta_0.$$



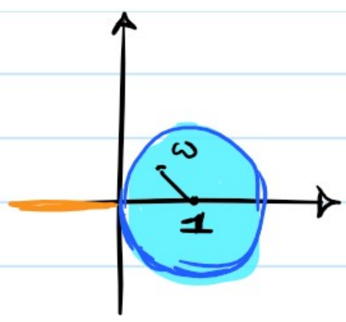
$$\boxed{z = e^{i\theta} \Rightarrow dz = i e^{i\theta}\, d\theta}$$

∎

$\underline{\text{Corollary}}$. $\mathrm{Ln}(z) = \ln|z| + i\,\mathrm{Arg}(z)$ for $z \in \mathbb{C} \setminus \mathbb{R}^{\leq 0}$.

$\underline{\text{Corollary}}$. $e^{\mathrm{Ln}(z)} = z$ for $z \in \mathbb{C} \setminus \mathbb{R}^{\leq 0}$.

Let $f_n(z) = 1 + z + \cdots + z^n$. So $f_n(z) \to \frac{1}{1-z}$ for $|z| < 1$ and uniformly on closed disks. So for any $\varepsilon > 0$ and $n \gg_\varepsilon 1$ for any z in segment of $[1, \omega]$ we have



$$\left| f_n(1-z) - \frac{1}{z} \right| < \varepsilon.$$

$$\left| f_n(1-z) - \frac{1}{z} \right| < \varepsilon.$$

So $\left| \int_1^\omega f_n(1-z)\, dz - \int_1^\omega \frac{dz}{z} \right| < \varepsilon$. Hence

$$\operatorname{Ln}(\omega) = \int_1^\omega \frac{dz}{z} \xleftarrow[n\to\infty]{} \int_1^\omega f_n(1-z)\, dz = -\int_0^{1-\omega} f_n(z)\, dz$$

$$= -\sum_{k=0}^{n} \frac{z^{k+1}}{k+1} \Big|_0^{1-\omega}$$

$$= -\left[(1-\omega) + \frac{(1-\omega)^2}{2} + \cdots + \frac{(1-\omega)^{n+1}}{n+1} \right]$$

$\Rightarrow \forall\ |z|<1,\ \operatorname{Ln}(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots$

<u>What happens for $|z|=1$?</u>

Let $S_n = e^{i\theta} + \frac{e^{2i\theta}}{2} + \cdots + \frac{e^{ni\theta}}{n}$ and $a_n = e^{i\theta} + \cdots + e^{ni\theta}$.

Then $S_n = a_1 + \frac{(a_2-a_1)}{2} + \frac{(a_3-a_2)}{3} + \cdots + \frac{(a_n - a_{n-1})}{n}$

$$= a_1\left(1-\tfrac{1}{2}\right) + a_2\left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \cdots + a_{n-1}\left(\tfrac{1}{n-1} - \tfrac{1}{n}\right) + \frac{a_n}{n}$$

$$= \frac{a_1}{1\times 2} + \frac{a_2}{2\times 3} + \cdots + \frac{a_{n-1}}{(n-1)\times n} + \frac{a_n}{n}.$$

$$a_n = e^{i\theta} \cdot \frac{e^{ni\theta}-1}{e^{i\theta}-1} \Rightarrow |a_n| \le \frac{2}{|e^{i\theta}-1|}.$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{|a_n|}{(n-1)n} \text{ is convergent}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{a_n}{(n-1)n} \text{ is absolutely convergent}$$

$$\text{and } \frac{a_n}{n} \xrightarrow[n\to\infty]{} 0$$

So $\{S_n\}_{n=1}^{\infty}$ is a convergent series $\Rightarrow \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$ is a convergent series.

By continuity of Ln and $\sum \frac{z^n}{n}$ we get

$$\operatorname{Ln}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for } |z-1| \le 1 \text{ and } z \ne 1.$$

<u>Proposition.</u> Suppose U is an open simply-connected region, $f: U \to \mathbb{C}$ is holomorphic and $0 \notin f(U)$.

$f: U \to \mathbb{C}$ is holomorphic and $0 \notin f(U)$.

Then $\exists\, g: U \to \mathbb{C}$, holomorphic and $e^{g(z)} = f(z)$.

(we denote it by $\log f(z)$.)

(Notice that g is NOT unique, for instance $g(z) + 2\pi i$ is another such function.)

<u>Proof.</u> For any $z_0 \in U$, let $g(z_0) = \log(f(a)) + \int_\gamma \frac{f'(z)}{f(z)}\, dz$,

where log is defined as above with respect to a ray which does not pass through $f(a)$, e.g. if $f(a) \notin \mathbb{R}^{\leq 0}$, we can take <u>Ln</u>, and γ is a piecewise diff. curve which starts from a and ends at z_0, and it is in U.

Let $\omega = f(z)$. So by chain rule,

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \int_{f\circ\gamma} \frac{d\omega}{\omega} \in \log f(z_0) - \log f(a) + 2\pi i \mathbb{Z}.$$

for a <u>log</u> defined as above for a ray which does NOT pass through $z_0, a, z_0 a^{-1}$.

$\Rightarrow e^{g(z_0)} = f(z_0)$. ■

<u>Remark</u>. As you have seen above, we can define different "complex logarithm maps" on various domains. Essentially the above argument implies: for any simply-connected open region U of $\mathbb{C}$, there is a holomorphic function $\log_U : U \to \mathbb{C}$ s.t.

$$e^{\log_U(z)} = z$$

for any $z \in U$.

Notice that, if U_1 and U_2 are two such regions, then

$\forall z \in U_1 \cap U_2$, $\log_{U_1} z = \log_{U} z \pmod{2\pi i \mathbb{Z}}$.

$\forall z \in U_1 \cap U_2, \quad \log_{U_1} z = \log_{U_2} z \pmod{2\pi i \mathbb{Z}}.$

So there is a map $\ln : \mathbb{C}^\times \to \mathbb{C}/_{2\pi i \mathbb{Z}}$ s.t.

① $\forall z \in \mathbb{C}^\times, \quad e^{\ln z} = z$

(Notice the exponential function factors through $\mathbb{C}/_{2\pi i \mathbb{Z}}$.)

② (In particular) $\forall z_1, z_2 \in \mathbb{C}^\times$,

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2.$$

③ $\forall U \subseteq \mathbb{C}^\times$ simply-connected,

$\exists \log_U : U \to \mathbb{C}$ a holomorphic map s.t.

$\forall z \in U, \quad \log_U z + 2\pi i \mathbb{Z} = \ln z.$

Two extremely important properties of holomorphic functions:

<u>Theorem</u> U: open, $D \subseteq U$ closed disk centered at z_0.

f: holomorphic on U

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad \text{for any } z \in \mathring{D}.$$

(Holomorphic $\Rightarrow$ analytic.)

<u>Theorem</u> U: open, connected

f_1, f_2: holomorphic on U

$\exists$ distinct points $z_1, z_2, \dots \in U$ s.t.

① $z_n \to z_0 \in U$

② $f_1(z_n) = f_2(z_n)$ for $n \in \mathbb{Z}^{\geq 0}$.

$\Rightarrow f_1(z) = f_2(z)$ for any $z \in U$.

(Holomorphic extensions are unique.)

I will not present proofs of these theorems in class, but I include their proofs here.

Proof of Holomorphic $\Rightarrow$ Analytic.

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{I}$$

$$|z - z_0| < |\zeta - z_0| \Rightarrow \left|\frac{z-z_0}{\zeta - z_0}\right| < 1$$

$$\Rightarrow \sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n = \frac{1}{1-\frac{z-z_0}{\zeta-z_0}} = \frac{\zeta - z_0}{\zeta - z}.$$

$$\Rightarrow \frac{1}{\zeta - z} = \sum_{n=0}^{\infty}\frac{1}{(\zeta-z_0)^{n+1}}(z-z_0)^n. \qquad \text{II}$$

$$\text{I}, \text{II} \Rightarrow f(z) = \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\, d\zeta$$

because of absol. conver

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta\right)(z-z_0)^n$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n. \qquad \blacksquare$$

Proof of uniqueness of holomorphic extension.

Considering $f = f_1 - f_2$. We need to show that, if $\{z \in U \mid f(z) = 0\}$ has a limit point and U is connected, then $f = 0$ on U.

Let O be the interior of $\{z \in U \mid f(z) = 0\}$.

Step 1. An open disk around z_0 is a subset of O. In particular, $O \neq \emptyset$.

Pf of step 1. If not, $\exists$ a disk $D \subseteq U$ around z_0 s.t. $f|_D \neq 0$.

By the previous theorem, $\forall z \in D$, $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Let n_0 be the smallest index where $a_{n_0} \neq 0$. Then

$$f(z) = a_{n_0}(z-z_0)^{n_0}\left(1 + g(z)\right)$$

where g is holomorphic on D and $g(z_0)=0$.

$\Rightarrow \quad 0=f(z_i)=a_{n_0}(z_i-z_0)^{n_0}(1+g(z_i))$ for any i.

Since $z_i\neq z_0$, $1+g(z_i)=0 \Rightarrow -1=g(z_i)\xrightarrow[i\to\infty]{} g(z_0)=0$

which is a contradiction.

Step 2. $\mathcal{O}$ is a closed subset of U.

Pf of step 2. Suppose $\omega_i\in\mathcal{O}$ and $\omega_i\to\omega_0\in U$.

Then $f(\omega_i)\to f(\omega_0)\Rightarrow f(\omega_0)=0$. Now by step 1 $\omega_0\in\mathcal{O}$.

. $\left.\begin{array}{l}\mathcal{O} \text{ is open, closed, and non-empty} \\ U \text{ is connected}\end{array}\right\}\Rightarrow \mathcal{O}=U.$ ∎