Name: ____________________

PID: ____________________

Section: ____________________

Question	Points	Score
1	16	
2	6	
3	6	
4	14	
5	18	
Total:	60	

1. Write your Name, PID, and Section on the front page of your exam.
2. No smart phone or other electronic devices are allowed during this exam.
3. If you have to leave the lecture hall during the exam for any reason, you have to leave your exam and smart phone or other electronic devices on the table in front of the lecture hall with the proctor.
4. Read each question carefully, and answer each question completely.
5. Write your solutions clearly in the exam sheet.
6. If you are using a result proved in lectures, you have to clearly write down the statement of this result.
7. Show all of your work; no credit will be given for unsupported answers.

1. (a) (2 points) Find the characteristic of $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_9$. Justify your answer.

(b) (2 points) True or False. There is an integral domain D such that

$$1_D + 1_D \neq 0 \text{ and } 1_D + 1_D + 1_D + 1_D = 0.$$

Justify your answer.

(c) (2 points) Find $(3^{-1})(4)$ in $\mathbb{Z}_7$. Justify your answer.

(d) (4 points) True or false. If I is a prime ideal of A and A/I is finite, then I is a maximal ideal. Justify your answer.

(e) (3 points) True or False. $x^8 + 3x^6 + 6x^4 + 5x^2 + 1$ has no zero in $\mathbb{Q}$.

(f) (3 points) Find all primes p such that $x - 1$ is a factor of $x^{10} + 2x^3 + x + 1$ in $\mathbb{Z}_p[x]$.

2. (a) (3 points) Suppose $x^7 - x + 3$ is irreducible in $\mathbb{Z}_7[x]$. Prove that

$$8x^7 + 21x^3 - 14x^2 - 22x + 10$$

is irreducible in $\mathbb{Q}[x]$.

(b) (3 points) Prove that $x^7 + 42x^3 - 30x^2 + 18x + 12$ is irreducible in $\mathbb{Q}[x]$.

3. Let $\alpha = \sqrt[5]{2}$ and

$$\mathbb{Q}[\alpha] = \{\sum_{i=0}^{n} a_i \alpha^i |\ a_i \in \mathbb{Q}, n \in \mathbb{Z}^+\}.$$

(a) (3 points) Find the minimal polynomial $m_\alpha(x)$ of α over $\mathbb{Q}$.

(b) (1 point) Prove that $\{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}[\alpha]$.

(c) (2 points) Write α^{-1} as a $\mathbb{Q}$-linear combination of $1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

4. Let $D := \mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} | a, b \in \mathbb{Z}\}$. We know that D is a subring of $\mathbb{C}$. Let $N : D \to \mathbb{Z}[\sqrt{-6}], N(a + b\sqrt{-6}) = a^2 + 6b^2$. We know that $N(z_1 z_2) = N(z_1)N(z_2)$.

   (a) (2 points) Prove that if $N(z) = 1$ for some $z \in D$, then $z = \pm 1$. Deduce that $D^\times = \{\pm 1\}$.

   (b) (2 points) Prove that there is no $z \in D$ such that $N(z) = 2$.

(c) (4 points) Prove that $\sqrt{-6}$ is irreducible in D.

(d) (4 points) Prove that $\sqrt{-6}$ is not prime in D.

(e) (2 points) Prove that D is not a UFD.

5. (a) (4 points) Prove that $x^3 - x + 1$ is irreducible in $\mathbb{Z}_3[x]$.

(b) (4 points) Suppose E is a splitting field of $x^3 - x + 1$ over $\mathbb{Z}_3$ and $\alpha \in E$ is a zero of $x^3 - x + 1$. Prove that

$$x^3 - x + 1 = (x - \alpha)(x - \alpha - 1)(x - \alpha - 2).$$

(c) (2 points) Use part (b) to prove that $E = \mathbb{Z}_3[\alpha]$.

(d) (2 points) Prove that $E = \{a_0 + a_1\alpha + a_2\alpha^2 | a_0, a_1, a_2 \in \mathbb{Z}_3\}$.

(e) (2 points) Prove that $|E| = 27$.

(f) (2 points) Prove that $\alpha^{27} - \alpha = 0$.

(g) (2 points) Prove that $x^3 - x + 1 | x^{27} - x$ in $\mathbb{Z}_3[x]$. (Hint. Think about the minimal polynomial of α over $\mathbb{Z}_3$.)

Good Luck!