Homework 7, math103b winter 2019

1. Suppose $D = \mathbb{Z}[\sqrt{-21}]$.

   (a) Prove that $D^{\times} = \{\pm 1\}$.

   (b) Prove that $\sqrt{-21}$ is irreducible in D.

   (c) Prove that $\sqrt{-21}$ is not prime in D.

   (d) Deduce that D is not a UFD.

2. Suppose E is a field and $\mathbb{Z}/p\mathbb{Z} \subseteq E$ where p is prime. Suppose $\alpha \in E$ is a zero of $x^p - x + 1$. Prove that

$$x^p - x + 1 = (x-\alpha)(x-\alpha-1)\cdots(x-\alpha-p+1)$$

   in $E[x]$. (**Hint**. Using Fermat's little theorem show that $\alpha + i$ is a zero of $x^p - x + 1$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. Then use the generalized factor theorem.)

3. Let $\beta := \sqrt[n]{2}$ where n is a positive integer.

   (a) Prove that the minimal polynomial $m_\beta(x)$ of β over $\mathbb{Q}$ is $x^n - 2$.

   (b) Prove that $\{1, \beta, \ldots, \beta^{n-1}\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}[\beta]$.

(c) Write β^{-1} as a $\mathbb{Q}$-linear combination of $1, \beta, \ldots, \beta^{n-1}$.