Homework 6, math103b winter 2019

1. Suppose E is a field extension of $\mathbb{Z}_3$ that contains a zero α of $x^3 - x + 1$.

   (a) Prove that $\mathbb{Z}_3[\alpha] = \{a_0 + a_1\alpha + a_2\alpha^2 | a_0, a_1, a_3 \in \mathbb{Z}_3\}$.

   (b) Prove that $\mathbb{Z}_3[\alpha]$ is a field and $|\mathbb{Z}_3[\alpha]| = 27$.

2. Let $I = \{2p(x) + xq(x) | \, p(x), q(x) \in \mathbb{Z}[x]\}$. Prove that I is not a principal ideal and deduce that $\mathbb{Z}[x]$ is not a PID.

3. Let $\beta := \sqrt{1 + \sqrt{3}}$.

   (a) Prove that the minimal polynomial $m_\beta(x)$ of β over $\mathbb{Q}$ is $x^4 - 2x^2 - 2$.

   (b) Prove that

   $$\mathbb{Q}[\beta] = \{a_0 + a_1\beta + a_2\beta^2 + a_3\beta^3 | \, a_0, a_1, a_2, a_3 \in \mathbb{Q}\}.$$

   (c) Prove that $\mathbb{Q}[\beta] \simeq \mathbb{Q}[x]/(x^4 - 2x^2 - 2)\mathbb{Q}[x]$ and it is a field.

   (d) Write β^{-1} as a $\mathbb{Q}$-linear combination of $1, \beta, \beta^2, \beta^3$.