Homework 5, math103b winter 2019

1. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$.

   (a) $x^n - 12$ where $n \in \mathbb{Z}^+$.

   (b) $x^3 - x^2 - x - 1$.

   (c) $x^5 - 10x^3 + 25x^2 - 51x + 2017$ (Only in this part of the problem you are allowed to use the following advance theorem: Let p be a prime and $a \in \mathbb{Z}_p \setminus \{0\}$. Then $x^p - x + a$ is irreducible in $\mathbb{Z}_p[x]$.)

2. Let $f_0(x) := x^5 - 3x^3 + 6x^2 + 9x - 21$.

   (a) Prove that $f_0(x)$ is irreducible in $\mathbb{Q}[x]$. (**Hint**; Think about a useful criterion!)

   (b) Let α be a real zero of $f_0(x)$. Suppose $\phi_\alpha : \mathbb{Q}[x] \to \mathbb{R}$ is the evaluation map at α; that means $\phi_\alpha(f(x)) := f(\alpha)$. Prove that

   $$\ker \phi_\alpha = f_0(x)\mathbb{Q}[x].$$

   (**Hint**: Use the fact that $\mathbb{Q}[x]$ is a PID and part (a).)