Homework 4, math103b winter 2019

1. Prove that $x^3 - x + 1 \in \mathbb{Z}_3[x]$ is irreducible.

2. Prove that $x^{2k} + x^{2k-1} + \cdots + x + 1$ has no zero in $\mathbb{Q}$ where k is a positive integer.

3. Suppose p is an odd prime. Prove that 2 is a zero of $x^{p^2+p} - x - 2$ in $\mathbb{Z}_p$.

4. Suppose p is an odd prime number. Notice that $\mathbb{Z}_p[x]$ is an integral domain of characteristic p.

   (a) Prove that $(x-1)^p = x^p - 1$ in $\mathbb{Z}_p[x]$.

   (b) Prove that $(x-1)^{p-1} = x^{p-1} + x^{p-2} + \cdots + 1$ in $\mathbb{Z}_p[x]$.

   (Hint: observe that $x^p - 1 = (x-1)(x^{p-1} + \cdots + x + 1)$; and use cancellation law in an integral domain.)

   (c) Use part (b) to deduce that for any $0 \le i \le p-1$ we have $\binom{p-1}{i} = (-1)^i$ in $\mathbb{Z}_p$.