# Homework 3, math103b winter 2019

1. Suppose $p$ is a prime and $a \in \mathbb{Z}_p \setminus \{0\}$. Show that $x^p - x + a$ has no zeros in $\mathbb{Z}_p$.

2. Find all the primes $p$ such that $x + 2$ is a factor of

$$x^6 - x^4 + x^3 - x + 1$$

in $\mathbb{Z}_p[x]$.

3. Find a zero of $x^3 - 2x + 1$ in $\mathbb{Z}_5[x]$ and express it as a product of a degree 1 and a degree 2 polynomial.

4. (a) In class we proved that, when $p$ is prime, for any $a \in \mathbb{Z}_p$ we have $a^p = a$. Use this result to show

$$x^p - x = x(x-1)\cdots(x-(p-1))$$

in $\mathbb{Z}_p[x]$.

(b) Use part (a) to deduce $(p-1)! = -1$ in $\mathbb{Z}_p$.

(Hint: think about zeros of $x^p - x$ in $\mathbb{Z}_p$.)

5. (a) Suppose $\omega := \frac{-1+\sqrt{-3}}{2}$. Show that

$$\mathbb{Z}[\omega] := \{a + b\omega \mid a, b \in \mathbb{Z}\}$$

is a subring of $\mathbb{C}$.

(b) Show that the field of fractions of $\mathbb{Z}[\omega]$ is

$$\mathbb{Q}[\omega] := \{a + b\omega \mid a, b \in \mathbb{Q}\}.$$

(Hint: Use $\omega^2 + \omega + 1 = 0$; observe that $\omega + \overline{\omega} = -1$ and $\omega\overline{\omega} = 1$ where $\overline{\omega}$ is the complex conjugate of $\omega$; compute $(a + b\omega)(a + b\overline{\omega})$.)

6. How many degree 2 and degree 3 polynomials are there in $\mathbb{Z}_2[x]$ that have no zeros in $\mathbb{Z}_2$?