Left cosets

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In the previous lecture we defined right cosets of a subgroup:

Suppose $(G, \cdot)$ is a group and H is a subgroup of G. Then

(1) for every $x \in G$, $Hx = \{hx \mid h \in H\}$

(2) for every $x, y \in G$, $Hx = Hy \iff xy^{-1} \in H$

(3) $H\backslash G := \{Hx \mid x \in G\}$ is a partition of G.

These statements were proved based on the fact that

$x \sim y \iff \exists h \in H, \; y = hx$ is an equivalent relation.

Similarly we can define left cosets of H and the set G/H of left cosets of H.

Proposition. Suppose $(G, \cdot)$ is a group and H is a subgroup of G. Let $x \sim' y \iff \exists h \in H, \; y = xh$. Then

(1) $\sim'$ is an equivalence relation.

(2) For every $x \in G$, $xH := \{xh \mid h \in H\}$ is the equivalent class of x under the relation $\sim'$.

(3) $G/H := \{xH \mid x \in G\}$ is a partition of G.

Pf. All the proofs are similar to the case of right cosets of H.

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Pf. (1) Reflexive. $x = x\cdot e_G \Rightarrow x \sim' x$.

Symmetric. $x \sim' y \Rightarrow \exists h\in H,\ x = yh \Rightarrow y = xh^{-1}$

$\Rightarrow y \sim' x$.

Transitive. $x \sim' y \Rightarrow \exists\, h_1 \in H,\ y = xh_1$
$y \sim' z \Rightarrow \exists\, h_2 \in H,\ z = yh_2$ $\Big\} \Rightarrow$

$z = y h_2 = (xh_1)h_2 = x\underbrace{(h_1h_2)}_{\text{in } H} \Rightarrow z \sim' x$.

(2) $[x]_{\sim'} = \{y\in G \mid y \sim' x\} = \{y \in G \mid x \sim' y\}$

$= \{y\in G \mid \exists h\in H,\ y = xh\}$

$= \{xh \mid h \in H\} = xH$.

(3) Since $\sim'$ is an equivalent relation, $\{[x]_{\sim'} \mid x\in G\}$ is a partition of G. Hence $\{xH \mid x\in G\}$ is a partition of G. ∎

When G is abelian, clearly the sets xH and Hx are the same. But when G is not abelian, these sets might be different. For instane consider $G = S_3$ and $H = \langle (1,2)\rangle$.

Then $(1,3)H = \{(1,3), (1,3)(1,2)\} = \{(1,3), (3,1,2)\}$

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and $H(1,3)=\{(1,3),(1,2)(1,3)\}=\{(1,3),(2,1,3)\}$.

Notice that $(3,1,2)\neq(2,1,3)$, and so

$$(1,3)H \neq H(1,3).$$

<u>Proposition</u>. Suppose G is a group and H is a subgroup of G. Let $N_G(H):=\{x\in G \mid xH=Hx\}$. Then $N_G(H)$ is a subgroup of G.

<u>Pf.</u>. We use the subgroup criterion. Notice that

$e_G H=\{e_G h \mid h\in H\}=\{h \mid h\in H\}=H$ and

$H e_G=\{h e_G \mid h\in H\}=\{h \mid h\in H\}=H$. Hence

$e_G H = H e_G$, which means $e_G\in N_G(H)$. Thus $N_G(H)$ is not empty.

- Suppose $x,y\in N_G(H)$. We want to show $xy^{-1}\in N_G(H)$.

This means we are assuming $xH=Hx$ and $yH=Hy$, and we want to show $(xy^{-1})H=H(xy^{-1})$.

Multiplying all elements of yH and Hy by y^{-1} from left and right, we conclude that $y^{-1}(yH)y^{-1}=y^{-1}(Hy)y^{-1}$. Hence

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$y^{-1}(yH)y^{-1} = y^{-1}(Hy)y^{-1} \Rightarrow Hy^{-1} = y^{-1}H.$ (I)

Multiplying both sides of (I) by x from left, we obtain

$$xHy^{-1} = xy^{-1}H. \qquad \text{(II)}$$

We also know $xH = Hx$. Multiplying all elements of these sets by y^{-1} from right, we deduce that

$$xHy^{-1} = Hxy^{-1}. \qquad \text{(III)}$$

By (II) and (III), we conclude that $xy^{-1}H = Hxy^{-1}$. Thus $xy^{-1} \in N_G(H)$. Hence by the subgroup criterion, $N_G(H)$ is a subgroup of G. ∎

Def. $N_G(H)$ is called the normalizer subgroup of H.

Def. H is called a normal subgroup if $N_G(H) = G$; that means, for every $x \in G$, $Hx = xH$. If H is a normal subgroup of G, we write $H \trianglelefteq G$.

Here is an alternative way of saying H is a normal subgroup.

Lemma. Suppose H is a subgroup of G. Then $H \trianglelefteq G$ if and only if $xHx^{-1} = H$ for every $x \in G$; that means conjugation

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by x sends H to itself.

Pf. If $H \trianglelefteq G$, then, for every $x \in G$, $xH = Hx$. Multiplying all the elements of these sets by x^{-1} from right, we obtain

$$xHx^{-1} = Hxx^{-1} = H.$$

If $xHx^{-1} = H$, multiplying all the elements of these sets by x from right, we obtain that $xHx^{-1}x = Hx$, and so $xH = Hx$. The claim follows. ■

Ex. If G is an abelian group, then every subgroup is normal.

Solution. If $H \le G$, then, for every $x \in G$,

$$xH = \{xh \mid h \in H\} = \{hx \mid h \in H\} = Hx. \quad ■$$

Ex. $\langle(1,2)\rangle$ is not a normal subgroup of S_3.

Solution. We have seen that $(1,3)\langle(1,2)\rangle \ne \langle(1,2)\rangle(1,3)$.

Ex. If $H \le G$ and $[G:H] = 2$, then $H \trianglelefteq G$.

Pf. Since $[G:H] = 2$, $H\backslash G := \{Hx \mid x \in G\}$ has exactly two elements. One of them is $H = He_G$, and suppose the other one is Hx_0. Therefore $\{H, Hx_0\}$ is a partition

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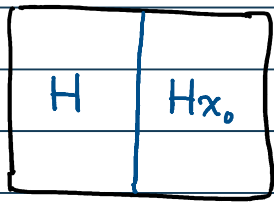
of G; that means $G = H \cup Hx_0$ and $H \cap Hx_0 = \emptyset$.

We want to show that, for every $x \in G$, $xH = Hx$.

For $x \in G$, there are two possibilities

either $x \in H$ or $x \in Hx_0$.



- If $x \in H$, then $xH = H$ and $Hx = H$; and so $xH = Hx$.
- If $x \in Hx_0$, then $x \notin H$ and $Hx_0 = Hx$. Hence

$\{H, Hx\}$ is a partition of G (I) and $H \neq xH$.

Since $\{yH \mid y \in G\}$ is a partition of G and $H \neq xH$, we have $H \cap xH = \emptyset$. By (I), the complement of H in G is Hx, and so $xH \subseteq Hx$. (II)

Since $x \notin H$, $x^{-1} \notin H$. Therefore by a similar argument as above, $x^{-1}H \subseteq Hx^{-1}$. Multiplying all the elements of these sets by x from left and right, we conclude that

$xx^{-1}Hx \subseteq xHx^{-1}x$; and so $Hx \subseteq xH$ (III)

By (II) and (III), $xH = Hx$. This completes the proof. ∎

Next we show that kernel of a group homomorphism is always

Kernel is normal

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normal

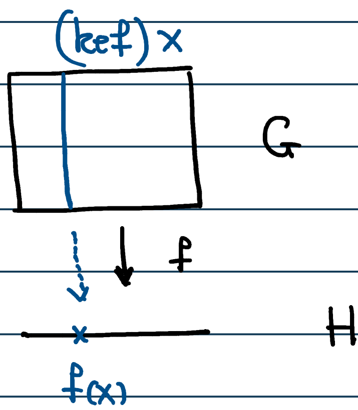
<u>Lemma</u>. Suppose $(G,\cdot)$ and $(H,*)$ are groups and $f: G \to H$ is a group homomorphism. Then

(1) for every $x \in G$,

$$(\ker f)\, x = \{y \in G \mid f(y) = f(x)\} \text{ and}$$

$$x\, (\ker f) = \{y \in G \mid f(y) = f(x)\}.$$

(2) $\ker f \trianglelefteq G$.



<u>Pf</u>. (1) $f(y) = f(x) \iff$

$f(x) * f(y)^{-1} = e_H \iff$

$f(x \cdot y^{-1}) = e_H \iff$

$x \cdot y^{-1} \in \ker f \iff$

$(\ker f)\, x = (\ker f)\, y \iff$

$y \in (\ker f)\, x$.

$f(y) = f(x) \iff f(y)^{-1} * f(x) = e_H \iff f(y^{-1} \cdot x) = e_H$

$\iff y^{-1} \cdot x \in \ker f \iff x\, (\ker f) = y\, (\ker f)$

$\iff y \in x\, (\ker f)$.

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(2) By part (1), for every $x \in G$, $(\ker f)x = x(\ker f)$, and so $\ker f \trianglelefteq G$. ∎

Let's recall that $\mathbb{Z}_n = \{[x]_n \mid x \in \mathbb{Z}\}$ and

$$[x]_n = \{x + nk \mid k \in \mathbb{Z}\}. \qquad \text{(I)}$$

Notice that $\{nk \mid k \in \mathbb{Z}\} = n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, and (I) implies that $[x]_n = x + n\mathbb{Z}$ is a left coset of $n\mathbb{Z}$. Hence $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the set of all the left cosets of $n\mathbb{Z}$. Let's recall that we defined $+$ (and $\cdot$) on $\mathbb{Z}_n$ equivalent class representatives: $[x]_n + [y]_n := [x+y]_n$, and we proved that this is well-defined and $(\mathbb{Z}_n, +)$ is a group.

<u>Question</u>. Can we define an operation on G/H in a similar way? This means: is $(xH)\cdot(yH) := xyH$ (*) well-defined?

Next we show that (*) is well-defined if H is a normal subgroup, and it is a good exercise to show the converse.

<u>Lemma</u>. Suppose $N \trianglelefteq G$. Then $(xN)\cdot(yN) := xyN$ is a well-defined operation on G/N.

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Pf. We have to show that if $xN = x'N$ and $yN = y'N$, then $xyN = x'y'N$. (To make sure that this operation does not depend on the choice of a representative from the left cosets xH and yH.)

$$(xN = x'N \text{ and } yN = y'N) \overset{?}{\Longrightarrow} (xy)N = (x'y')N$$

$$\Updownarrow$$

$$(xy)^{-1}(x'y') \in N$$

$$(xy)^{-1}(x'y') = y^{-1}x^{-1}x'y' = y^{-1}n_1 y' = y^{-1}y'n_2$$

$$\boxed{xN = x'N \Rightarrow x^{-1}x' = n_1 \in N}$$

$$\boxed{\begin{array}{l} n_1 y' \in Ny' = y'N \quad \text{because } N \trianglelefteq G \\ \Rightarrow n_1 y' = y'n_2 \text{ for some } n_2 \in N. \end{array}}$$

$$\boxed{yN = y'N \Rightarrow y^{-1}y' = n_3 \in N}$$

$$\| \quad n_3 n_2$$

Therefore $(xy)^{-1}(x'y') \in N$, which implies

$$(xy)N = (x'y')N. \qquad \blacksquare$$

This takes us to the definition of a factor group.

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Proposition. Suppose $(G, \cdot)$ is a group and $N \trianglelefteq G$. Then

(1) $(xN)\cdot(yN) := (xy)N$ is a well-define operation on G/N

(2) $(G/N, \cdot)$ is a group with the neutral element N.

(3) $p: G \to G/N$, $p(x) := xN$ is a surjective group homomorphism. (p is called the natural quotient map.)

(4) $\ker p = N$.

Pf. Part (1) is proved in the previous lemma.

(2) Neutral element. $(e_G N)\cdot(xN) = (e_G x)N = xN$

$$(xN)\cdot(e_G N) = (xe_G)N = xN$$

(Notice that $e_G N = N$.)

Inverse. $(x^{-1}N)\cdot(xN) = (x^{-1}x)N = e_G N = N$

$$(xN)\cdot(x^{-1}N) = (xx^{-1})N = e_G N = N$$

Associative. $(xN \cdot yN)\cdot zN = (xy)N \cdot zN = ((xy)z)N$

$$\| $$

$$xN\cdot(yN\cdot zN) = xN\cdot (yz)N = (x(yz))N$$

(We are doing the computations in G and "decorate" them with N.)

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(3). $p(x)\cdot p(y) = xN \cdot yN = (xy)N = p(xy)$ for every $x,y\in G$.

Hence p is a group homomorphism.

- $x \in \ker p \iff p(x) =$ neutral element of G/N

$$\iff xN = e_G N$$

$$\iff e_G^{-1} x \in N \iff x \in N$$

Hence $\ker p = N$.

Ex Prove that $A_n \trianglelefteq S_n$ and write a multiplication table of S_n/A_n for $n \geq 2$.

Pf. $\mathrm{sgn} : S_n \to \{1,-1\}$ is a group homomorphism and $\ker(\mathrm{sgn}) = A_n$. Hence $A_n \trianglelefteq S_n$ (kernel of a group homomorphism is a normal subgroup)

We have seen that $A_n \backslash S_n = \{A_n, A_n(1,2)\}$. So $S_n/A_n = \{A_n, (1,2)A_n\}$.

$\cdot$	A_n 1	$(1,2)A_n$ -1
1 A_n	A_n 1	$(1,2)A_n$ -1
-1 $(1,2)A_n$	$(1,2)A_n$ -1	A_n 1

Observe that

$A_n \mapsto 1$, $(1,2)A_n \mapsto -1$

is an isomorphism $S_n/A_n \longrightarrow \{1,-1\}$.

Fundamental theorem of group homomorphisms

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The following theorem is crucial in understanding the group structure of image of group homomorphism and factor groups.

Fundamental Theorem of Group Homomorphisms (also known as the first isomorphism theorem)

Suppose $(G,\cdot)$ and $(H,*)$ are two groups and $f: G \to H$ is a group homomorphism. Then

(1) $\ker f \trianglelefteq G$ and $\operatorname{Im} f \leq H$.

(2) $\bar{f}: G/\ker f \longrightarrow \operatorname{Im} f$, $\bar{f}(x \ker f) := f(x)$ is a a well-defined group isomorphism.

Pf. We have already proved part (1). To show part (2), we start by showing why $\bar{f}$ is well-defined.

Independence of the choice of a coset representative:

$$x \ker f = x' \ker f \overset{?}{\Longrightarrow} f(x) = f(x').$$

$$x \ker f = x' \ker f \Rightarrow x^{-1} \cdot x' \in \ker f$$

$$\Rightarrow f(x^{-1} \cdot x') = e_H$$

$$\Rightarrow f(x)^{-1} * f(x') = e_H \Rightarrow f(x') = f(x).$$

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$\bar{f}(x \ker f)$ belongs to the codomain

$\bar{f}(x \ker f) = f(x) \in \operatorname{Im} f$.

$\bar{f}$ is a group homomorphism.

$$\begin{aligned}
\bar{f}((x \ker f)\cdot(y \ker f)) &= \bar{f}((xy)\ker f) \\
&= f(xy) \\
&= f(x) * f(y) \\
&= \bar{f}(x \ker f) * \bar{f}(y \ker f).
\end{aligned}$$

$\bar{f}$ is injective.

$\bar{f}(x \ker f) = \bar{f}(y \ker f) \overset{?}{\Rightarrow} x \ker f = y \ker f.$

$$\begin{aligned}
\bar{f}(x \ker f) = \bar{f}(y \ker f) &\Rightarrow f(x) = f(y) \\
&\Rightarrow f(y)^{-1} * f(x) = e_H \\
&\Rightarrow f(y^{-1}x) = e_H \\
&\Rightarrow y^{-1}x \in \ker f \\
&\Rightarrow x \ker f = y \ker f.
\end{aligned}$$

codomain of $\bar{f}$

$\bar{f}$ is surjective. $z \in \operatorname{Im} f \Rightarrow z = f(x)$ for some $x \in G$

$\Rightarrow z = \bar{f}(x \ker f) \in \operatorname{Im} \bar{f}$. Hence $\bar{f}$ is surjective $\blacksquare$

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Ex. Prove that $\mathbb{R}/\mathbb{Z} \simeq S^1$ where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Pf. Earlier we have seen that

$$f: \mathbb{R} \to S^1, \quad f(x) = e^{2\pi i x}$$

is a group homomorphism, $\ker f = \mathbb{Z}$, and $\operatorname{Im} f = S^1$. Hence by the 1st isomorphism theorem,

$$\bar{f}: \mathbb{R}/\mathbb{Z} \to S^1, \quad \bar{f}(x + \mathbb{Z}) := e^{2\pi i x}$$

is a group isomorphism. ∎

Ex. $(\mathbb{C} \setminus \{0\})/S^1 \simeq \mathbb{R}^+$

Pf. Earlier we have seen that $N: \mathbb{C} \setminus \{0\} \to \mathbb{R}^+$, $N(z) = |z|$ is a surjective group homomorphism. Notice that $z \in \ker N \Leftrightarrow |z| = 1 \Leftrightarrow z \in S^1$. Therefore by the 1st isomorphism theorem, $(\mathbb{C} \setminus \{0\})/\ker N \simeq \operatorname{Im} N$ which means

$(\mathbb{C} \setminus \{0\})/S^1 \simeq \mathbb{R}^+$. ∎

Ex. Suppose $G = \langle g \rangle$ is a cyclic group of order m. Then $\mathbb{Z}_m \simeq G$.

Pf. We have seen one proof of this statement before

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Here we are going to provide another proof.

We have seen that $f: \mathbb{Z} \to G$, $f(n) = g^n$ is a group homomorphism. Since $G = \langle g \rangle$, f is surjective. We have seen that $|\langle g \rangle| = o(g) = m$. Hence

$$\begin{array}{ccc} g^k = e_G & \Leftrightarrow & m \mid k \\ \Updownarrow & & \\ f(k) = e_G & & \end{array}$$

$$k \in \ker f \Leftrightarrow f(k) = e_G.$$

Thus $\ker f = m\mathbb{Z}$. By the 1st isomorphism theorem, we conclude that $\mathbb{Z}/\ker f \simeq \operatorname{Im} f$, and so $\mathbb{Z}/m\mathbb{Z} \simeq G$. Notice that $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m$. Therefore $\mathbb{Z}_m \simeq G$. ∎

<u>Ex.</u> (Chinese Remainder Theorem) Suppose $m, n \in \mathbb{Z}^+$ and $\gcd(m,n) = 1$. Then $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$.

<u>Pf.</u> (Another method was suggested in your HW assignment.)

Let $f: \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$ be $f(a) := ([a]_m, [a]_n)$. Then for every $a, b \in \mathbb{Z}$, $f(a+b) = ([a+b]_m, [a+b]_n) =$

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for every $a, b \in \mathbb{Z}$, $f(a+b) = ([a+b]_m, [a+b]_n)$

$$= ([a]_m + [b]_m, [a]_n + [b]_n)$$

$$= ([a]_m, [a]_n) + ([b]_m, [b]_n)$$

$$= f(a) + f(b).$$

Hence f is a group homomorphism. Next we find its kernel:

$a \in \ker f \iff f(a) = ([0]_m, [0]_n)$

$$\iff [a]_m = [0]_m \text{ and } [a]_n = [0]_n$$

$$\iff m \mid a \quad \text{and} \quad n \mid a$$

$$\iff mn \mid a \qquad (\text{since } \gcd(m,n) = 1)$$

Therefore $\ker f = mn\mathbb{Z}$. By the 1st isomorphism theorem, $\mathbb{Z}/\ker f \simeq \operatorname{Im} f$. Thus $\mathbb{Z}/mn\mathbb{Z} \simeq \operatorname{Im} f$. (I) Notice that $\mathbb{Z}/mn\mathbb{Z} = \mathbb{Z}_{mn}$ (II), and so $|\operatorname{Im} f| = mn$. Because $\operatorname{Im} f \leq \mathbb{Z}_m \times \mathbb{Z}_n$, $|\operatorname{Im} f| = mn$, and $|\mathbb{Z}_m \times \mathbb{Z}_n| = mn$, we conclude that $\operatorname{Im} f = \mathbb{Z}_m \times \mathbb{Z}_n$ (III). By (I), (II), and (III), we have $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$. ∎

Let's finish with a non-abelian example.

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<u>Ex.</u> Let $G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R}\setminus\{0\},\ b \in \mathbb{R} \right\}$ and

$N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}$. Then

(1) $G \leq GL_2(\mathbb{R})$ (2-by-2 invertible matrices)

(2) $N \trianglelefteq G$ and $G/N \simeq \mathbb{R}\setminus\{0\}$.

<u>Pf.</u> (1) We use the subgroup criterion. Notice that

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$. Next we find $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1}$.

(the general rule: $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}^{-1} = \frac{1}{\det.} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$.)

<u>Claim1</u> $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{bmatrix}$.

<u>Pf of Claim1.</u> $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{bmatrix} =$

$\begin{bmatrix} a \cdot a^{-1} + b \cdot 0 & -a \cdot a^{-1}b + b \cdot 1 \\ 0 \cdot a^{-1} + 1 \cdot 0 & -0 \cdot a^{-1}b + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

Hence $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} \in G$.

<u>Claim 2.</u> $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} aa' & ab'+b \\ 0 & 1 \end{bmatrix} \in G$.

<u>Pf of Claim 2.</u> $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \overbrace{aa'}^{\neq 0} & \overbrace{ab'+b}^{\text{in } \mathbb{R}} \\ 0 & 1 \end{bmatrix} \in G$.

By Claim 1 and 2, $\forall\, x, y \in G$, $xy^{-1} \in G$. Thus G is a subgroup of $GL_2(\mathbb{R})$.

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(2) Let $f: G \to \mathbb{R}\setminus\{0\}$, $f\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) := a$. Then by Claim 2,

$$f\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} aa' & ab'+b \\ 0 & 1 \end{bmatrix}\right)$$

$$= aa'$$

$$= f\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) f\left(\begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix}\right)$$

Hence f is a group homomorphism.

- $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in \ker f \iff f\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) = 1 \iff a = 1.$

  Hence $\ker f = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\} = N$. Therefore $N \trianglelefteq G$.

- $\forall a \in \mathbb{R}\setminus\{0\}$, $a = f\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right) \in \operatorname{Im} f$. Thus $\operatorname{Im} f = \mathbb{R}\setminus\{0\}$.

- By the 1st isomorphism theorem,

$$G/\ker f \simeq \operatorname{Im} f \implies G/N \simeq \mathbb{R}\setminus\{0\}. \quad \blacksquare$$