Greatest common divisor

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To understand what elements of $\mathbb{Z}_n$ have multiplicative inverse, we need to recall basic properties of greatest common divisor of integers. In particular, we recall Euclid's algorithm.

The greatest common divisor of two non-zero integers a and b is, as its name suggests,

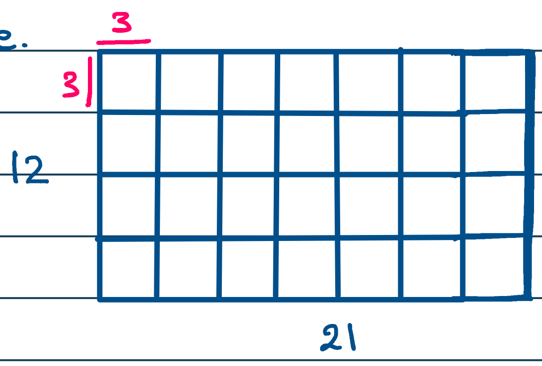
$$\max\{d \in \mathbb{Z} \mid d \mid a, \ d \mid b\},$$

and it is denoted by $\gcd(a,b)$.

Notice that if a is a non-zero integer and $d \mid a$, then $d \leq |a|$. Hence $\gcd(a,b) \leq \min\{|a|,|b|\}$ if a, b are two non-zero integers.

Pictorially $\gcd(a,b)$ is the size of the largest square tile which can cover an $a \times b$ rectangle.

To find this tile, each time we cut the largest possible from one of the edges. We stop when we get a square!



Basic of divisibility

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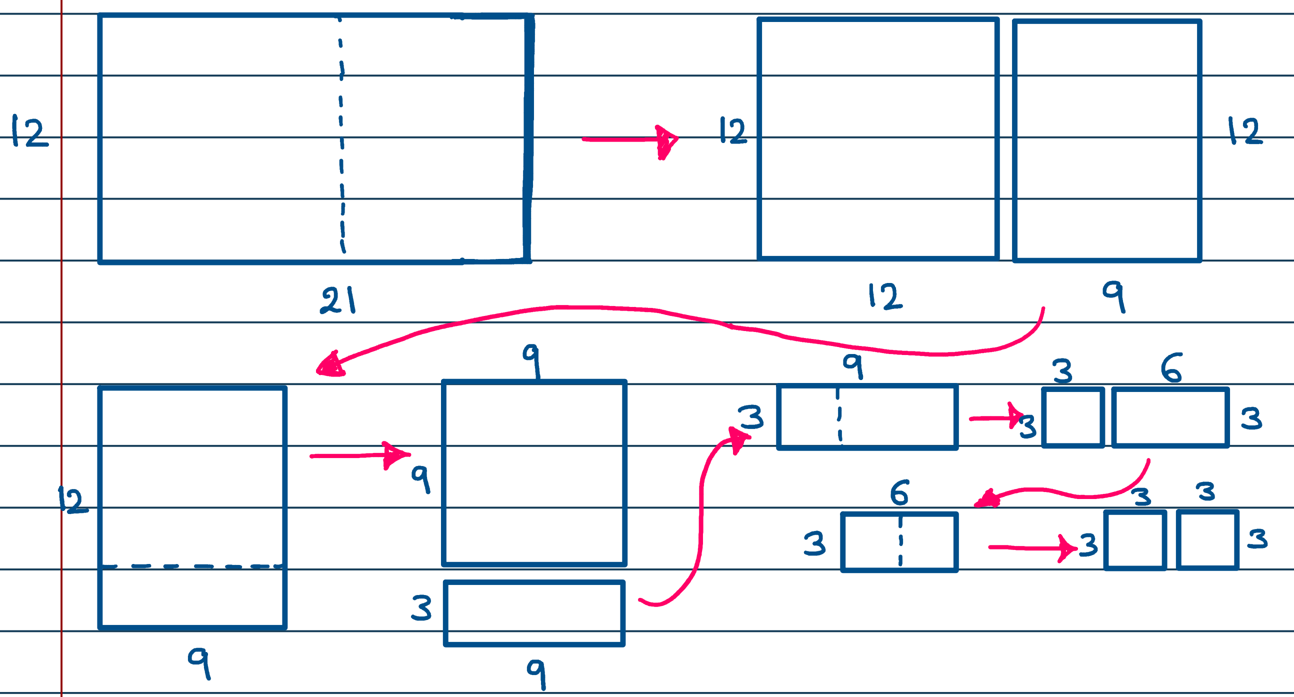

The above process is essentially Euclid's algorithm. We will make the process more formal and prove why it works.

<u>Lemma</u>. Suppose $a, b, d \in \mathbb{Z}$. Then

- $d \mid a,\ d \mid b \Rightarrow d \mid ra + sb$ for every $r, s \in \mathbb{Z}$
- $d \mid b,\ d \mid a-b \Rightarrow d \mid a$

<u>Pf.</u>
- $\left.\begin{array}{l} d \mid a \Rightarrow a = kd \text{ for some } k \in \mathbb{Z} \\ d \mid b \Rightarrow b = ld \text{ for some } l \in \mathbb{Z} \end{array}\right\} \Rightarrow$

  $ra + sb = rkd + sld = \underbrace{(rk + sl)}_{\text{in } \mathbb{Z}} d \Rightarrow d \mid ra + sb.$
- $d \mid b,\ d \mid a-b \Rightarrow d \mid (1)(b) + (1)(a-b) \Rightarrow d \mid a.$ ■

The following is a corollary of the above lemma.

Steps of Euclid's algorithm

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Corollary. Suppose a, b are two positive integers. Then

$$\gcd(a,b) = \gcd(b, a-b)$$

Pf. We show that d is a common divisor of a and b if and only if d is a common divisor of b and $a-b$.

$d \mid a,\ d \mid b \Rightarrow d \mid (1)(a) + (-1)(b) \Rightarrow d \mid a-b.$

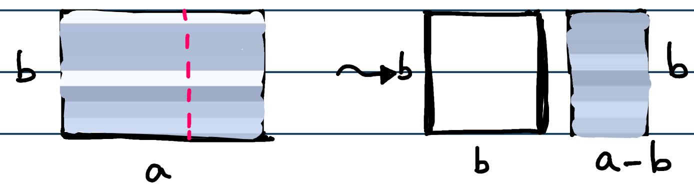
Hence $d \mid b$ and $d \mid a-b$.

If $d \mid b$ and $d \mid a-b$, then $d \mid a$ by the previous lemma.

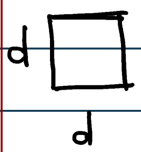
Hence $\gcd(a,b) = \max\{d \in \mathbb{Z} \mid d \mid a,\ d \mid b\}$

$$= \max\{d \in \mathbb{Z} \mid d \mid b,\ d \mid a-b\} = \gcd(b, a-b). \qquad \blacksquare$$

The above corollary justifies the steps in the pictorial argument:



By cutting, we do not change the gcd of the sides. So repeating this process we end up getting a square



, and $\gcd(d,d) = d$, which means sides of this square is the gcd of the sides of the initial rectangle.

Next we point out the connection with Euclid's algorithm.

Euclid's algorithm

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__Lemma__. Suppose n is a non-zero integer. If $a \overset{n}{\equiv} a'$, then

$$\gcd(a,n) = \gcd(a',n).$$

__Pf.__ Since $a \overset{n}{\equiv} a'$, $a - a' = nk$ for some $k \in \mathbb{Z}$.

- $d \mid n$ and $d \mid a' \Rightarrow d \mid (k)n + (1)a' \Rightarrow d \mid a$. (I)
- $d \mid n$ and $d \mid a \Rightarrow d \mid (1)a + (-k)n \Rightarrow d \mid a'$. (II)

By (I), (II), $\{d \in \mathbb{Z} \mid d \mid n, d \mid a\} = \{d \in \mathbb{Z} \mid d \mid n, d \mid a'\}$. Hence $\gcd(n,a) = \gcd(n,a')$. ∎

Euclid's algorithm is a fast way of finding the gcd of two positive integers. Similar to the pictorial method, Euclid's algorithm gives us a process through which the gcd stays the same, but we get smaller and smaller pairs.

Suppose $a \geq b$ are two positive integers. Let $a_0 := a$, $a_1 := b$. We divide a_0 by a_1:

$a_0 = a_1 \cdot q_1 + a_2$. Then $a_0 \overset{a_1}{\equiv} a_2$, and so by the above lemma, $\gcd(a_0, a_1) = \gcd(a_1, a_2)$. Next we divide a_1 by a_2 if $a_2 \neq 0$, and repeat this process till the remainder is 0.

Euclid's algorithm

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$a_0 = a_1 q_1 + a_2$, $\gcd(a_0, a_1) = \gcd(a_1, a_2)$, $a_1 > a_2$

$a_1 = a_2 q_2 + a_3$, $\gcd(a_1, a_2) = \gcd(a_2, a_3)$, $a_2 > a_3$

$\vdots \qquad \vdots \qquad \vdots$

$a_{n-1} = a_n q_n + 0$, $\gcd(a_{n-1}, a_n) = a_n$, $a_n > 0$

Hence $a_0 \geq a_1 > a_2 > \cdots > a_n > 0$ and $a_n = \gcd(a_0, a_1) = \gcd(a, b)$.

Notice that, for every $0 \leq i < n$,

$$\begin{bmatrix} 0 & 1 \\ 1 & -q_i \end{bmatrix} \begin{bmatrix} a_{i-1} \\ a_i \end{bmatrix} = \begin{bmatrix} (0)(a_{i-1}) + (1)(a_i) \\ (1)(a_{i-1}) + (-q_i)(a_i) \end{bmatrix} = \begin{bmatrix} a_i \\ a_{i-1} - a_i q_i \end{bmatrix} = \begin{bmatrix} a_i \\ a_{i+1} \end{bmatrix}.$$

Hence $$\begin{bmatrix} a_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -q_n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -q_{n-1} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & -q_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}.$$

Therefore $a_n = r a_0 + s a_1$ for some integers r, s. (*)

The following theorem follows.

<u>Theorem</u>. For every non-zero integers a and b, there are integers r and s such that $\gcd(a, b) = ra + sb$.

<u>Pf</u>. We notice that $\gcd(a, b) = \gcd(|a|, |b|)$. Now claim follows from (*). ∎

<u>Remark</u>. (*) gives us an algorithm to find an integer solution for $\gcd(a, b) = ax + by$.

Euclid's algorithm: an example

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<u>Ex.</u> Find gcd(197, 79) and write it as an integer linear combination of 197 and 79.

<u>Solution.</u> We follow Euclid's algorithm. Let $a_0 = 179$, $a_1 = 79$.

$197 = 79 \times 2 + 39$, $\quad q_1 = 2$, $\quad a_2 = 39$

$79 = 39 \times 2 + 1$, $\quad q_2 = 2$, $\quad a_3 = 1$

$39 = 1 \times 39 + 0$, $\quad q_3 = 39$, $\quad a_4 = 0$

So $\gcd(197, 79) = 1$ and

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -q_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -q_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -q_1 \end{bmatrix} \begin{bmatrix} 197 \\ 79 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -39 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -39 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ * & * \end{bmatrix}$$

This means $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ * & * \end{bmatrix} \begin{bmatrix} 197 \\ 79 \end{bmatrix}$. Comparing the 1st components, we obtain that

$$1 = 197 \times (-2) + 79 \times (5). \qquad \blacksquare$$

Basic properties of gcd

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Here we review basic properties of gcd of two integers.

<u>Theorem</u>. Suppose a, b are two non-zero integers. Then if $\gcd(a,b)=d$, then $\gcd(\frac{a}{d}, \frac{b}{d})=1$.

<u>Pf</u>. Since $\gcd(a,b)=d$, $d \mid a$ and $d \mid b$ and

$$d = ra + bs \quad \text{for some } r,s\in\mathbb{Z}.$$

Hence $\frac{a}{d}, \frac{b}{d} \in \mathbb{Z}$ and $1 = r(\frac{a}{d}) + s(\frac{b}{d})$. (I)

Let $d' := \gcd(\frac{a}{d}, \frac{b}{d})$. Then $d' \mid \frac{a}{d}$, $d' \mid \frac{b}{d}$, and so $d' \mid r(\frac{a}{d}) + s(\frac{b}{d})$ (II). Therefore by (I) and (II), $d' \mid 1$.

Thus $d'=1$, which means $\gcd(\frac{a}{d}, \frac{b}{d})=1$. ∎

<u>Theorem</u>. Suppose a, b are two non-zero integers. If $d := \gcd(a,b)$ and d' is a common divisor of a and b, then $d' \mid d$.

<u>Pf</u>. Since $d = \gcd(a,b)$, $d = ra + sb$ for some $r,s\in\mathbb{Z}$.

Because $d' \mid a$ and $d' \mid b$, $d' \mid ra+sb$. Hence $d' \mid d$. ∎

<u>Theorem</u>. Suppose a,b,c are three non-zero integers. Then $\gcd(ac, bc) = |c| \gcd(a,b)$.

<u>Pf</u>. Suppose $d := \gcd(a,b)$. Then $d \mid a$ and $d \mid b$, and so

Basic properties of gcd

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$d|c|$ divides ac and $d|c|$ divides bc. Hence

$$d|c| \leq \gcd(ac, bc). \qquad \text{(I)}$$

On the other hand, $d = \gcd(a,b)$ implies that $d = ra + sb$ for some $r, s \in \mathbb{Z}$. Hence

$$d|c| = ra\,|c| + sb\,|c| = \pm\,(r(ac) + s(ac)). \qquad \text{(II)}$$

Notice that $\gcd(ac, bc)$ divides every integer linear combination of ac and bc. Hence by (II),

$$\gcd(ac, bc) \mid d|c|. \qquad \text{(III)}$$

By (I) and (III), $\gcd(ac, bc) = d|c|$, which means

$$\gcd(ac, bc) = |c| \gcd(a,b). \qquad ∎$$

Euclid's lemma. For $a, b, c \in \mathbb{Z} \setminus \{0\}$,

$$\left.\begin{array}{l} \gcd(a,b) = 1 \\ a \mid bc \end{array}\right\} \Rightarrow a \mid c.$$

(If a and b do not have any non-trivial common factors and a divides bc, then $a \mid c$. This lemma plays an important role in proving the uniqueness of factorization of integers into prime numbers.)

Euclid's lemma

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Pf. Since $\gcd(a,b)=1$, $\underbrace{1=ra+bs}_{(I)}$ for some $r,s\in\mathbb{Z}$.

Multiplying both sides of (I) by c, we obtain that

$$c = rc(a) + s(bc). \qquad \text{(II)}$$

Since $a|a$ and $a|bc$, a divides every integer linear combination of a and bc. Therefore by (II), $a|c$. ∎

An important corollary of Euclid's lemma is about prime numbers. Let's recall that an integer p is called prime if $p>1$ and p has exactly two positive divisors 1 and p.

This means if p is prime, $d\in\mathbb{Z}^+$, and $d|p$, then $d=1$ or p.

Here is an important corollary of Euclid's lemma.

Euclid's lemma: prime number case Suppose p is prime and a,b are two non-zero integers. Then

$$p|ab \text{ implies that either } p|a \text{ or } p|b.$$

Pf. If $p\nmid a$, then $\gcd(p,a)\neq p$. Hence $\gcd(p,a)=1$ as p has exactly two positive divisors 1 and p.

Since $\gcd(p,a)=1$ and $p|ab$, by Euclid's lemma, $p|b$. ∎