Introduction

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. Historically algebra was created to understand zeros of polynomial equations. Along the way the importance of various system of numbers and their symmetries became evident. The importance of symmetries of objects in other parts of math and other sciences turned it into an important part of algebra which has connections with geometry, analysis, combinatorics, topology, etc. Symmetries of objects are studied in group theory, which is the main subject of our course.

. We start by recalling some of the basic concepts from set theory and congruence arithmetic.

Equivalence Relation. Let X be a non-empty set. A relation over X is a subset R of $X \times X$. If $(x,y) \in R$, we say x is R-related to y and write $x R y$. Suppose R is a relation over X. Then: . R is called reflexive if $\forall x \in X$, xRx.

. R is called symmetric if $\forall\, x,y \in X$, $x R y \Rightarrow y R x$.

. R is called transitive if $\forall x,y,z \in X$, $\left.\begin{matrix} xRy \\ yRz \end{matrix}\right\} \Rightarrow xRz$.

Equivalent relation

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. R is called an equivalent relation if R is reflexive, symmetric, transitive.

Equivalent relations are essentially about equality with respect to certain measurment. The following example is an important indication of this concept:

Ex. Suppose X and Y are two non-empty sets and $f: X \to Y$ is a function. Let $\sim$ be the following relation over X:

$$\forall x_1, x_2 \in X, \quad x_1 \sim x_2 \iff f(x_1) = f(x_2).$$

Then $\sim$ is an equivalent relation.

Pf. Reflexive. $\forall x \in X$, $f(x) = f(x) \Rightarrow x \sim x$.

Symmetric. $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2) \Rightarrow f(x_2) = f(x_1)$

$$\Rightarrow x_2 \sim x_1.$$

Transitive. $$\left.\begin{array}{l} x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2) \\ x_2 \sim x_3 \Rightarrow f(x_2) = f(x_3) \end{array}\right\} \Rightarrow f(x_1) = f(x_3) \Rightarrow x_1 \sim x_3.$$

■

Alternatively equivalent relations partion X into subsets that share the same property. For instance in the previous example

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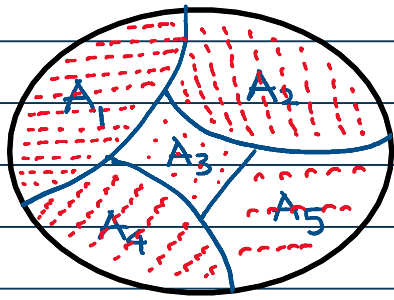
the shared property is having the same value under the function f.

Let's recall that $\mathcal{P}$ is called a partition of a non-empty set X,

if
- $\mathcal{P}$ consists of non-empty subsets of X, (subsets)
- $A, B \in \mathcal{P}$ and $A \neq B \Rightarrow A \cap B = \emptyset$ (disjoint)
- $\forall x \in X, \exists A \in \mathcal{P}, x \in A$ (Covering)

(Alternatively, $\bigcup_{A \in \mathcal{P}} A = X$.)

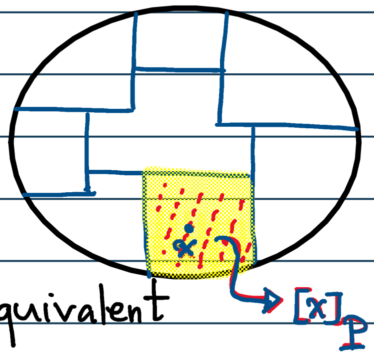


Suppose $\mathcal{P}$ is a partition of X. Then we get a classification function from X to $\mathcal{P}$: $X \to \mathcal{P}$, $x \mapsto [x]_{\mathcal{P}}$ where $[x]_{\mathcal{P}}$ is the unique element of $\mathcal{P}$ which contains x. Notice that because of the covering condition x is contained in some element of $\mathcal{P}$, and because of the disjointness condition x is in a unique element of $\mathcal{P}$. By the previous example, $x \sim_{\mathcal{P}} y \Leftrightarrow [x]_{\mathcal{P}} = [y]_{\mathcal{P}}$ is an equivalent relation. So we obtain the following lemma:

<u>Lemma</u>. Suppose $\mathcal{P}$ is a partition of a non-empty set X. For $x, y \in X$, $x \sim y$ if x and y are in the same element of $\mathcal{P}$.

Then $\sim$ is an equivalent relation.

<u>Pf.</u> For $x \in X$, let $[x]_P$ be the unique element of P which contains x. So $x \mapsto [x]_P$ is a function from X to P. By the previous example, $x \sim y \Leftrightarrow [x]_P = [y]_P$ is an equivalent relation over X. Notice that this means $x \sim y$ exactly when x and y are in the same element of P. ■



Starting with an equivalent relation $\sim$ over a non-empty set X, we can partition X with respect to $\sim$, as we show next.

For $x \in X$, let $[x] := \{y \in X \mid y \sim x\}$ (all the elements that are $\sim$-related to x.) We call $[x]$ the equivalent class of x with respect to $\sim$. When $x \sim y$, we say x is equivalent to y with respect to $\sim$.

<u>Proposition</u>. Suppose $\sim$ is an equivalent relation over a non-empty set X. Then $\{[x] \mid x \in X\}$ is a partition of X.

<u>Lemma</u>. $x \sim y \Longleftrightarrow [x] = [y]$.

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<u>Pf of Lemma</u>. ($\Leftarrow$) We want to show $[x]=[y] \Rightarrow x\sim y$.

$$\left.\begin{array}{r} x\sim x \Rightarrow x\in[x] \\ [x]=[y] \end{array}\right\} \Rightarrow x\in[y] \Rightarrow x\sim y.$$

($\Rightarrow$) WTS $x\sim y \Rightarrow [x]=[y]$. To show equality of sets $[x]$ and $[y]$, it is necessary and sufficient to prove $[x]\subseteq[y]$ and $[y]\subseteq[x]$. Let's start by proving $[x]\subseteq[y]$.

$$\left.\begin{array}{r} z\in[x] \Rightarrow z\sim x \\ x\sim y \end{array}\right\} \Rightarrow z\sim y \Rightarrow z\in[y].$$

Hence $[x]\subseteq[y]$. This means we showed

$$x\sim y \Rightarrow [x]\subseteq[y]. \qquad \text{(I)}$$

Notice that $x\sim y \Rightarrow y\sim x$. Therefore, by (I), $[y]\subseteq[x]$.

$$x\sim y \Rightarrow y\sim x \Rightarrow [y]\subseteq[x] \qquad \text{(II)}$$

By (I) and (II), we have $x\sim y \Rightarrow [x]=[y]$. ∎

<u>Pf of Proposition</u>. $\forall x\in X$, $x\sim x$. Thus $x\in[x]$. This implies that $[x]$'s are non-empty subsets and they cover X.

Next we show the disjointness property. Suppose $z\in[x]\cap[y]$.

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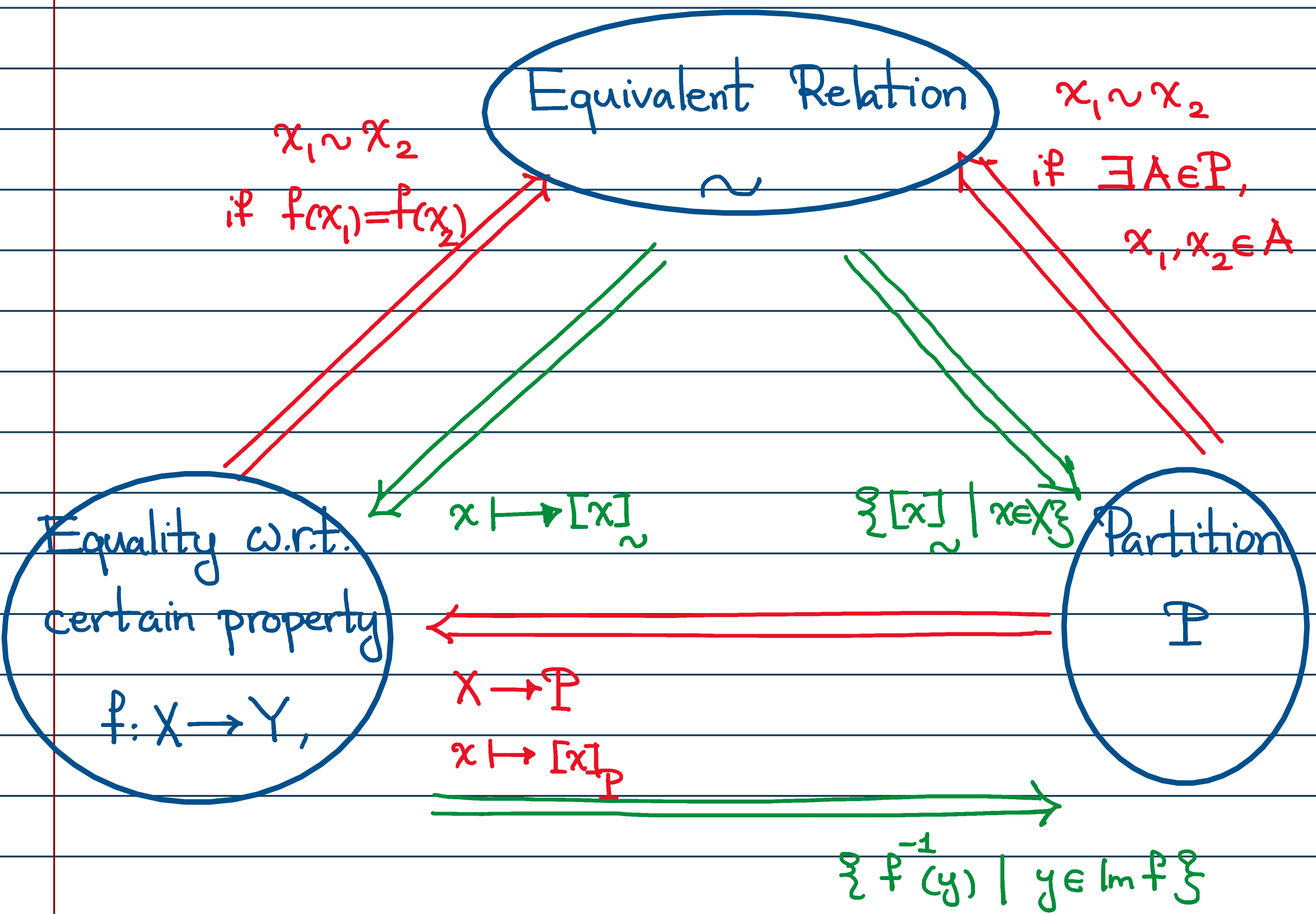
$$\left.\begin{array}{l} z\in[x] \Rightarrow z\sim x \Rightarrow [z]=[x] \\ z\in[y] \Rightarrow z\sim y \Rightarrow [z]=[y] \end{array}\right\} \Rightarrow [x]=[y].$$

We showed that $[x]\cap[y]\neq\emptyset \Rightarrow [x]=[y]$. The contrapositive of this statement is

$$[x]\neq[y] \Rightarrow [x]\cap[y]=\emptyset,$$

which is the disjointness property. ∎



Next we recall the congruence modulo n relation which plays an important role in our course.