Fifth problem set

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1. Prove the following isomorphisms:

(a) $\mathbb{Z}[x]/_{n\mathbb{Z}[x]} \simeq \mathbb{Z}_n[x]$ for any $n \in \mathbb{Z}^{\geq 2}$.

(b) $\mathbb{Q}[x]/_{\langle (x+1)(x-2)\rangle} \simeq \mathbb{Q}[x]/_{\langle x+1\rangle} \oplus \mathbb{Q}[x]/_{\langle x-2\rangle}$

(Hint. Consider $f(x) \longmapsto (f(x)+\langle x+1\rangle, f(x)+\langle x-2\rangle)$;

You might find $\frac{1}{3}(x+1)$ and $\frac{-1}{3}(x-2)$ useful!)

(c) $\mathbb{Q}[x]/_{\langle x^2-2x+6\rangle} \simeq \{c_0+c_1A \mid c_0, c_1 \in \mathbb{Q}\}$

where $A = \begin{bmatrix} 0 & -6 \\ 1 & 2 \end{bmatrix}$.

(So the RHS $= \left\{ \begin{bmatrix} c_0 & -6c_1 \\ c_1 & c_0+2c_1 \end{bmatrix} \middle| c_0, c_1 \in \mathbb{Q} \right\}$.)

(Hint. Consider the evalution at A, $\phi_A : \mathbb{Q}[x] \longrightarrow M_2(\mathbb{Q})$.)

2. Let $I = \langle 2, x\rangle$ in $\mathbb{Z}[x]$. Prove that I is NOT a principal ideal. (So $\mathbb{Z}[x]$ is NOT a PID.)

3. (a) Show that $\sqrt{-10}$ is irreducible in $\mathbb{Z}[\sqrt{-10}]$

(You can use without proof that $\mathbb{Z}[\sqrt{-10}] = \{a+b\sqrt{-10} \mid a, b \in \mathbb{Z}\}$ is a ring.)

(Hint. If $\sqrt{-10} = (a+\sqrt{-10}\,b)(c+\sqrt{-10}\,d)$, then their norm as complex numbers are equal $\Rightarrow$ $10 = (a^2+10b^2)(c^2+10d^2)$.)

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(b) Show that $5 \times 2 \in \langle \sqrt{-10} \rangle$ and $5, 2 \notin \langle \sqrt{-10} \rangle$.

And conclude that $\langle \sqrt{-10} \rangle$ is not prime.

(c) Prove that $\mathbb{Z}[\sqrt{-10}]$ is NOT a PID.

(Hint. Look for a result with the keywords:

PID, irreducible, maximal.)

4. (a) Prove that $\mathbb{Q}[x]/\langle x^4 - 2x^2 - 2 \rangle \simeq \{c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 \mid c_i \in \mathbb{Q}\}$

where $\alpha = \sqrt{1+\sqrt{3}} \in \mathbb{R}$.

(b) Write α^{-1} in the form $c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3$ with $c_i \in \mathbb{Q}$.

5. Let E be a field extension of $\mathbb{Z}_3$, which contains a zero α of $x^3 - x + 1$.

(a) Prove that $\mathbb{Z}_3[\alpha] = \{c_0 + c_1\alpha + c_2\alpha^2 \mid c_i \in \mathbb{Z}_3\}$ is a field.

(b) Find the number of elements of $\mathbb{Z}_3[\alpha]$.