Fourth problem set

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1. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$.

(a) x^3-3x^2+3x+4.

(b) x^n-12 where $n\in\mathbb{Z}^+$.

(c) $x^5-10x^3+25x^2-51x+2017$

(Only in this part of the problem you are allowed to use the following (advance) theorem:

Let p be a prime and $a\in\mathbb{Z}_p\setminus\{0\}$. Then x^p-x+a is irreducible in $\mathbb{Z}_p[x]$.)

2. (a) Prove that $f_0(x)=x^5-3x^3+6x^2+9x-21$ is irreducible in $\mathbb{Q}[x]$. (Hint. Think about a useful criterion!)

(b) Let α be a real zero of $f_0(x)$. Suppose $\phi_\alpha:\mathbb{Q}[x]\longrightarrow\mathbb{R}$ is the evaluation homomorphism; that means $\phi_\alpha(f(x))=f(\alpha)$. Prove that

$$\ker\phi_\alpha=\langle f_0(x)\rangle.$$

(Hint. Use the fact that $\mathbb{Q}[x]$ is a PID and part (a)).

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3. (a) Prove that $\left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \middle| \, a,b \in \mathbb{Q} \right\}$ is a subring of $M_2(\mathbb{Q})$.

(b) Prove that $f: \mathbb{Q}[\sqrt{2}] \longrightarrow \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \middle| \, a,b \in \mathbb{Q} \right\}$,

$$f(a+b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$$

is a ring isomorphism.