Third problem set

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1. In class we proved that, for any $a \in \mathbb{Z}_p$, we have $a^p = a$. (where p is prime). Use this result to show

$$x^p - x = x(x-1)\cdot\cdots\cdot(x-(p-1))$$

in $\mathbb{Z}_p[x]$. Use this result to deduce $(p-1)! = -1$ in $\mathbb{Z}_p$.
(Hint. Think about zeros of $x^p - x$ in $\mathbb{Z}_p$.)

2. (a) Show that $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$ where $\omega = \frac{-1+\sqrt{-3}}{2}$.

(b) Show that the field of fractions of $\mathbb{Z}[\omega]$ is

$$\mathbb{Q}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Q}\}.$$

(Hint. Use $\omega^2 + \omega + 1 = 0$; and compute $(a+b\omega)(a+b\bar{\omega})$ where $\bar{\omega} = \frac{-1-\sqrt{-3}}{2}$. (Notice $\omega + \bar{\omega} = -1$ and $\omega\bar{\omega} = 1$.))

3. Find all the primes p such that $x+2$ is a factor of $x^6 - x^4 + x^3 - x + 1$ in $\mathbb{Z}_p[x]$.

4. Find a zero of $x^3 - 2x + 1$ in $\mathbb{Z}_5$ and express it as a product of a degree 1 and a degree 2 polynomial.

5. How many degree 2 and degree 3 polynomials with no zeros in $\mathbb{Z}_2[x]$ are there?