QUIZ 3, MATH100C, SPRING 2021

1. Suppose A and B are finite abelian groups and $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. Let $f : A \times B \to S^1$ be a pairing; this means that f is a group homomorphism with respect to each component separately. Let
$$f_A : A \to \widehat{B},\ (f_A(a))(b) := f(a,b) \quad \text{and} \quad f^B : B \to \widehat{A},\ (f^B(b))(a) := f(a,b),$$
where $\widehat{A}$ and $\widehat{B}$ are the duals of A and B, respectively. Suppose f_A is a group isomorphism.
   (a) (2 points) Suppose $b \in \ker f^B$. Prove that $b \in \ker \chi$ for every $\chi \in \widehat{B}$.
   (b) (2 points) Prove that f^B is injective.
   (c) (2 points) Prove that f^B is an isomorphism.
   (You are allowed to use results about the dual of finite abelian groups that are proved in the lectures, except the above problem!)

2. Suppose F is a field of characteristic 0 which contains an element ζ of order n and $\overline{\mathrm{F}}$ is an algebraic closure of F. Suppose $E \in \mathrm{Int}(\overline{\mathrm{F}}/F)$, E/F is a Galois extension, and $\mathrm{Aut}_F(E)$ is a finite abelian group of exponent n. Let
$$\Delta(E) := ((E^\times)^n \cap F^\times)/F^{\times n}.$$
   (a) (2 points) Define the Kummer pairing $f : \mathrm{Aut}_F(E) \times \Delta(E) \to M_n$ where $M_n := \{1, \zeta, \ldots, \zeta^{n-1}\}$.
   (b) (3 points) Prove that the Kummer pairing is well-defined.
   (c) (3 point) Prove that $f^{\Delta(E)} : \Delta(E) \to \widehat{\mathrm{Aut}_F(E)}$ is injective where $f^{\Delta(E)}$ is given as in the first problem.
   (d) (2 points) Prove that $|\Delta(E)| \le |\mathrm{Aut}_F(E)|$.
   (This problem consists of some steps in the proof of the finite abelian case of Kummer theory. It goes without saying that you are not allowed to use the final result of Kummer theory for this problem.)

3. Suppose p is a prime and $F := \mathbb{Q}[\zeta_p]$ where $\zeta_p := e^{\frac{2\pi i}{p}}$. Suppose $a_1, \ldots, a_n \in F^\times$ are such that
$$\overline{A} := \langle a_1(F^{\times p}), \ldots, a_n(F^{\times p})\rangle$$
is a group of order p^n. Let $E := F[\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_n}]$ where $\sqrt[p]{a_i}$ is a zero of $x^p - a_i$ in $\overline{\mathrm{F}}$.
   (a) (3 points) Let $g : \mathbb{Z}_p^n \to \overline{A}$, $g([m_1]_p, \ldots, [m_n]_p) := (\prod_{i=1}^n a_i^{m_i})(F^{\times p})$. Prove that g is a well-defined isomorphism.
   (b) (3 points) Suppose $K \in \mathrm{Int}(E/F)$ and $[K : F] = p$. Prove that K/F is Galois and $\mathrm{Aut}_F(K) \simeq \mathbb{Z}_p$.
   (c) (2 points) Suppose $K \in \mathrm{Int}(E/F)$ and $[K : F] = p$. Prove that $K = F[\sqrt[p]{a}]$ for some $a \in F$, where $\sqrt[p]{a}$ is a zero of $x^p - a$ in $\overline{\mathrm{F}}$.
   (d) (1 point) Suppose $K \in \mathrm{Int}(E/F)$ and $K = F[\sqrt[p]{a}]$ for some $a \in F$, where $\sqrt[p]{a}$ is a zero of $x^p - a$ in $\overline{\mathrm{F}}$. Prove that $a(F^{\times p}) \in \Delta(E)$, where $\Delta(E)$ is given by Kummer theory (see problem 2).
   (e) (2 points) Suppose $K \in \mathrm{Int}(E/F)$ and $[K : F] = p$. Prove that $K = F[\sqrt[p]{a}]$ for some $a(F^{\times p}) \in \overline{A}$ that has order p.

(f) (2 points) Prove that there is a bijection between $\{K \in \mathrm{Int}(E/F) \mid [K : F] = p\}$ and one dimensional subspaces of $\mathbb{Z}_p^n$.

(g) (1 point) Prove that $|\{K \in \mathrm{Int}(E/F) \mid [K : F] = p\}| = \frac{p^n-1}{p-1}$.

(In this problem, you are allowed to use whatever result is proved in the lectures.)