QUIZ 5, VERSION A, MATH100B, WINTER 2021

Carefully state theorems that you are using.

1. Suppose F is a field and $\zeta \in F$ has multiplicative order n where n is an integer more than 1; that means $\zeta^n = 1$ and $\zeta^d \neq 1$ for every positive integer $d < n$. Suppose $a \in F$ and $x^n - a$ is irreducible in $F[x]$. Suppose E is a field extension of F which contains a zero $\sqrt[n]{a}$ of $x^n - a$.
   (a) (5 points) Prove that $F[\sqrt[n]{a}]$ is a splitting field of $x^n - a$ over F.

   (b) (2 points) Prove that $[F[\sqrt[n]{a}] : F] = n$.

   (c) (5 points) Prove that $\mathrm{Aut}_F(F[\sqrt[n]{a}]) \simeq \mathbb{Z}_n$.

2. (5 points) Suppose E is a field extension of F and $\alpha \in E$. Suppose $\gcd([F[\alpha] : F], 6) = 1$. Prove that $F[\alpha] = F[\alpha^3 - 2\alpha + 3]$.

3. Suppose F is a field, $f(x) \in F[x]$ is irreducible, and E is a splitting field of $f(x)$ over F. Suppose there is $\alpha \in E$ such that
$$f(\alpha) = f(\alpha + 1) = 0.$$

   (a) (3 points) Prove that there is $\theta \in \mathrm{Aut}_F(F[\alpha])$ such that $\theta(\alpha) = \alpha + 1$.

   (b) (2 points) Prove that there is a prime p such that $\mathrm{char}(F) = p$.

   (c) (5 points) Suppose $R := \{\alpha_1, \ldots, \alpha_n\}$ is the set of zeros of f in E. Let S_R be the group of all the permutations of R (the symmetric group of the set R). Prove that
$$r : \mathrm{Aut}_F(E) \to S_R, \;\; r(\theta) := \theta|_R$$
is a well-defined injective group homomorphism.

   (d) (3 points) Prove that there is $\widehat{\theta} \in \mathrm{Aut}_F(E)$ such that $r(\widehat{\theta})$ has $(\alpha, \alpha+1, \ldots, \alpha+p-1)$ in its cycle decomposition; that means
$$\widehat{\theta}(\alpha) = \alpha + 1, \; \widehat{\theta}(\alpha+1) = \alpha + 2, \; \ldots, \; \widehat{\theta}(\alpha+p-1) = \alpha.$$

   (e) (2 bonus points) Prove that $\mathrm{Aut}_F(E)$ has an element of order p.