QUIZ 4, VERSION A, MATH100B, WINTER 2021

1. (5 points) Prove that $\mathbb{Q}[\zeta_n]$ is a splitting field of $x^n - 1$ over $\mathbb{Q}$ where $\zeta_n := e^{2\pi i/n} \in \mathbb{C}$.

2. (5 points) Suppose F is a field, $f(x) \in F[x]$ is irreducible, and $f'(x) \neq 0$. Prove that $f(x)$ does not have multiple zeros in its splitting field over F.

3. Suppose F is a field and $f(x) \in F[x]$ is irreducible. Let E be a splitting field of f over F. Suppose $\alpha_1, \ldots, \alpha_n$ are all the distinct zeros of f in E.
   (a) (5 points) Suppose $\theta : F[\alpha_1] \to E$ is a ring homomorphism and $\theta(c) = c$ for every $c \in F$. Prove that $\theta(\alpha_1) = \alpha_i$ for some i.

   (b) (5 points) For every i, prove that there exists a unique ring homomorphism $\theta_i : F[\alpha_1] \to E$ such that $\theta_i(c) = c$ for every $c \in F$ and $\theta_i(\alpha_1) = \alpha_i$.

4. (10 points) Suppose E is a splitting field of $x^{31} - 1$ over $\mathbb{Z}_5$. Prove that $E \simeq \mathbb{F}_{125}$. (In this question, you are allowed to use the fact that the group of units of a finite field is cyclic and other results about finite fields that are proved in class.)