QUIZ 1, VERSION A, MATH100B, WINTER 2021

1. Answer the following questions and briefly justify your answers.
   (a) (1 point) True or false. Every integral domain can be embedded into a field.
   (b) (2 point) Find $|(\mathbb{Z}[x])^\times|$.
   (c) (3 points) True or false. There is an integral domain D such that
   $$\underbrace{1_D + \cdots + 1_D}_{9 \text{ times}} = 0 \text{ and } 1_D + 1_D + 1_D \neq 0.$$
   (d) (4 points) Find $|(\mathbb{Z}_9 \times \mathbb{Z}_5)^\times|$.

2. (5 points) Prove that $\mathbb{Q}[x]/\langle x^2 - 3\rangle \simeq \mathbb{Q}[\sqrt{3}]$ where $\mathbb{Q}[\sqrt{3}]$ is the smallest subring of $\mathbb{C}$ that contains $\mathbb{Q}$ and $\sqrt{3}$.

3. (5 points) Suppose p is a prime number and $f(x) \in \mathbb{Z}_p[x]$ is a polynomial of degree 3. Use the long division for polynomials to prove that $|\mathbb{Z}_p[x]/\langle f(x)\rangle| = p^3$.

4. Suppose m and n are positive integers and $\gcd(m, n) = 1$. Let $e : \mathbb{Z} \to \mathbb{Z}_n \times \mathbb{Z}_m,\ e(k) := k([1]_n, [1]_m)$. You can use without proof that e is a ring homomorphism.
   (a) (3 points) Find the kernel of e.
   (b) (4 points) Prove that e is surjective.
   (c) (3 points) Prove that $\mathbb{Z}/mn\mathbb{Z} \simeq \mathbb{Z}_n \times \mathbb{Z}_m$.