Math 100A - Fall 2019 - Midterm II

Name: ______________________________

Student ID: ___________________________

Instructions:

Please print your name, student ID.

During the test, you may not use books, calculators or telephones. *You may use any results proved in class, unless you are asked to prove these facts directly. If you use a homework problem, you would have to prove it.*

Read each question carefully, and show all your work. Disorganized writing or answers with no explanation will receive no credit, even if they are correct. *Please write your proofs neatly.*

There are 4 questions which are worth 40 points total. You have 80 minutes to complete the test.

Question	Score	Maximum
1		10
2		10
3		10
4		10
Extra credit		10
Total		40+10

Problem 1. [*10 points; 2, 2, 3, 3.*]

Let $G = \langle g \rangle$ be a cyclic group of order 20.

(i) List all other generators of G (expressing the answer as powers of g).

(ii) List all automorphisms $f : G \to G$.

(iii) List all subgroups of G.

(iv) Determine all elements of G of order 4 (justify for your answer).

Problem 2. [*10 points; 3, 2, 2, 3.*]

Let $f : G \to H$ be a group homomorphism, and let $K = \text{Ker } f = \{g : f(g) = 1\}$.

(i) Prove that K is a subgroup of G. This fact was shown in class, but you are asked to reprove it here.

(ii) Prove that for all $g \in G$, we have $gKg^{-1} \subset K$.

(iii) Conclude that K is normal in G.

(iv) Let $G = GL_2(\mathbb{R})$ be the group of 2×2 invertible real matrices. Using (iii), give an example of a *normal* subgroup H of G, with $H \neq \{1\}, H \neq G$.

Problem 3. [*10 points; 5, 5.*]

Let $G = \langle g \rangle$ and $H = \langle h \rangle$ be cyclic groups of finite orders m and n.

(i) If $\gcd(m, n) = 1$, show that $(g, h) \in G \times H$ is an element of order mn in $G \times H$. Conclude that $G \times H$ is cyclic.

(ii) If $\gcd(m, n) = d \neq 1$ show that $G \times H$ is not cyclic.

Problem 4. [*10 points; 3, 4, 3.*]

(i) Show that if $\sigma \in S_n$ satisfies $\sigma^3 = \epsilon$, then σ is a product of disjoint cycles of length 3. This is a particular case of a homework problem, but you are asked to reprove it.

(ii) Let G be a group. For each $a \in G$, let

$$\sigma_a : G \to G,\ \sigma_a(g) = aga^{-1}$$

be the associated inner automorphism. Let

$$f : G \to \mathrm{Inn}(G),\ \ a \mapsto \sigma_a.$$

We have seen in class that f is a homomorphism. Show that the kernel of f equals the center of G.

(iii) Show that for $n \geq 3$, the group $\mathrm{Aut}(S_n)$ has an element of order exactly 3.

Hint: The ideas in (i) and (ii) should play a role here.

Extra credit. [*10 points.*]

Find all subgroups of $(\mathbb{Z}, +)$.