SECOND MIDTERM
MATH 100A, UCSD, AUTUMN 16

You have 50 minutes.

There are 4 problems, and the total number of points is 60. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Problem	Points	Score
1	15	
2	15	
3	10	
4	20	
5	10	
6	10	
Total	60	

1. (15pts) *Give the definition of conjugation.*

The conjugate of h by g is ghg^{-1}.

(ii) *Give the definition of an isomorphism.*

A function $\phi\colon G \longrightarrow H$ between two groups is an isomorphism if ϕ is a bijection and $\phi(xy) = \phi(x)\phi(y)$ for all x and $y \in G$.

(iii) *The kernel of a group homomorphism.*

If $\phi\colon G \longrightarrow H$ is a group homomorphism the kernel of ϕ is the inverse image of the identity.

2. (15pts)
(i) *Find the cycle decomposition of the following element of* S_9,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 5 & 7 & 1 & 2 & 6 & 9 & 8 \end{pmatrix}.$$

$$(1,4,7,6,2,3,5)(8,9).$$

(ii) Compute the conjugate of σ by τ, where $\sigma = (1,5,6)(4,3,7,2)$ and $\tau = (1,5)(6,3,2)$.

$$(5,1,3)(4,2,7,6).$$

(iii) *Is it possible to conjugate* σ *to* σ', *where* $\sigma = (1,5,4)(2,3)$ *and* $\sigma' = (1,5)(3,4,2)$? *If so, find an element* τ *so that* σ' *is the conjugate of* σ *by* τ. *Otherwise explain why it is impossible.*

Yes it is possible, as both have the same cycle type.

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$$

will do the trick.

3. (10pts) *Let G be a group and let Z be its centre. Prove that if G/Z is cyclic, then G is abelian.*

Suppose that G/Z is generated by aZ. Then the elements of G/Z are of the form $a^i Z$, for $i \in \mathbb{Z}$.
Suppose that x and $y \in G$. Then xZ and yZ are two left cosets, so that $xZ = a^i Z$ and $yZ = a^j Z$, for some i and j. It follows that we may find z_1 and $z_2 \in Z$ so that $x = a^i z_1$ and $y = a^j z_2$.
We have

$$\begin{aligned} xy &= (a^i z_1)(a^j z_2) \\ &= a^i (z_1 a^j) z_2 \\ &= a^i (a^j z_1) z_2 \\ &= a^i a^j (z_1 z_2) \\ &= a^{i+j} (z_1 z_2). \end{aligned}$$

Similarly $yx = a^{j+i} z_2 z_1 = a^{i+j} z_1 z_2 = xy$. Thus G is abelian.

4. (20pts) (i) *Let $a \in G$. Prove that the map*

$$\sigma = \sigma_a \colon G \longrightarrow G \qquad \text{given as} \qquad \sigma(g) = aga^{-1},$$

is an automorphism of G.

Suppose that g and h are elements of G. We have

$$\begin{aligned} \sigma(g)\sigma(h) &= (aga^{-1})(aha^{-1}) \\ &= ag(a^{-1}a)ha^{-1} \\ &= agha^{-1} \\ &= \sigma(gh). \end{aligned}$$

Thus σ is a group homomorphism.

(ii) *Let $\phi \colon G \longrightarrow A(G)$ be the map which sends a to $\phi(a) = \sigma_a$. Show that ϕ is a group homomorphism.*

Let a and $b \in G$. Let $\sigma = \sigma_a$, $\tau = \sigma_b$ and $\rho = \sigma_{ab}$. We want to check that $\rho = \sigma\tau$. Both sides of this equation are functions from G to G, so we just need to check that they have the same effect on an element $g \in G$:

$$\begin{aligned} (\sigma\tau)(g) &= \sigma(\tau(g)) \\ &= \sigma(bgb^{-1}) \\ &= a(bgb^{-1})a^{-1} \\ &= (ab)g(b^{-1}a^{-1}) \\ &= (ab)g(ab)^{-1} \\ &= \rho(g). \end{aligned}$$

Thus ϕ is a group homomorphism.

(iii) *Show that the image $H = \phi(G)$ is isomorphic to G/Z, where Z is the centre of G.*

We check that Z is the kernel of ϕ. Suppose that $a \in Z$ and let $\sigma = \sigma_a = \phi(a)$. If $g \in G$ then

$$\sigma(g) = aga^{-1} = gaa^{-1} = g.$$

Thus σ is the identity map and so $a \in \operatorname{Ker}\phi$.
Now suppose that $a \in \operatorname{Ker}\phi$. Then σ is the identity map and so

$$g = \sigma(g) = aga^{-1}.$$

Multiplying on the right by a we get

$$ga = ag,$$

so that $a \in Z$. Thus $Z = \operatorname{Ker}\phi$ and the result follows by the first isomorphism theorem.

(iv) *Show that H is normal in* $\operatorname{Aut}(G)$.

Suppose that τ is an automorphism of G and let $\sigma = \sigma_a = \phi(a)$. Let $b = \tau(a)$ and let $\rho = \sigma_b = \phi(b)$. We check that

$$\tau\sigma\tau^{-1} = \rho.$$

Since both sides of this equation are functions from G to G we just need to check they have the same effect on elements g of G. As τ is a bijection we may find $h \in G$ such that $\tau(h) = g$. We have

$$\begin{aligned}
\tau\sigma\tau^{-1}(g) &= \tau\sigma\tau^{-1}(\tau(g)) \\
&= \tau(\sigma(h)) \\
&= \tau(aha^{-1}) \\
&= \tau(a)\tau(h)\tau(a^{-1}) \\
&= b\tau(h)b^{-1} \\
&= \rho(g).
\end{aligned}$$

As τ is arbitrary and $\rho \in H$ it follows that H is normal in $\operatorname{Aut}(G)$.

Bonus Challenge Problems

5. (10pts) *Prove the Third isomorphism theorem.*

Suppose that $K \subset H$ are two normal subgroups of a group G. Then

$$G/K \simeq \frac{G/H}{H/K}.$$

Let $\phi\colon G \longrightarrow G/H$ be the natural map. As the kernel of ϕ contains K there is a group homomorphism $f\colon G/K \longrightarrow G/H$ which makes the following diagram commute,

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "u"'] & G/H \\ G/K \arrow[ur, "f"'] & \end{tikzcd}$$

Since the diagram commutes, it follows that $f(gK) = gH$. It is then clear that f is surjective.

Suppose that gK is in the kernel of f.

$$f(gK) = gH = eH = H.$$

Thus $g \in H$. Thus the kernel consists of all left cosets of the form hK where $h \in H$, which is nothing more than H/K. Thus the result follows by the first isomorphism theorem.

6. (10pts) *Determine* Aut($\mathbb{Q}$), *the automorphism group of the rationals under addition.*

Aut($\mathbb{Q}$) is isomorphic to $\mathbb{Q}^*$ the non-zero rationals under multiplication. Define a function

$$f\colon \operatorname{Aut}(\mathbb{Q}) \longrightarrow \mathbb{Q}^*$$

by sending the automorphism ϕ to the image of 1, $f(\phi) = \phi(1)$. We check that f is a group homomorphism. Suppose that ϕ and ψ are two automorphisms of $\mathbb{Q}$ and let $a = \phi(1)$ and $b = \psi(1)$.

Then

$$\begin{aligned} f(\psi\phi) &= \psi(\phi(1)) \\ &= \psi(a) \\ &= a\psi(1) \\ &= ab \\ &= \phi(1)\psi(1) \\ &= f(\phi)f(\psi), \end{aligned}$$

where we got from the second line to the third line using the fact that ψ is an automorphism.

Thus f is a group homomorphism.

Suppose that ϕ belongs to the kernel of f. Then $f(\phi) = 1$ so that $\phi(1) = 1$. In this case

$$\phi(a) = a\phi(1) = a.$$

so that ϕ is the identity. It follows that f is injective.

Finally suppose that $a \in \mathbb{Q}^*$. Let

$$\phi\colon \mathbb{Q} \longrightarrow \mathbb{Q},$$

be the function which sends q to aq. We check that ϕ is an automorphism. The inverse map is the function which sends q to q/a. Thus ϕ is a bijection. We check that ϕ is a group homomorphism:

$$\begin{aligned} \phi(p+q) &= a(p+q) \\ &= ap + aq \\ &= \phi(p) + \phi(q). \end{aligned}$$

Thus $\phi \in \operatorname{Aut}(\mathbb{Q})$. But $f(\phi) = \phi(1) = a$, so that f is surjective.

Thus f is an isomorphism.