Math 100A - Fall 2019 - Midterm I

Name: ____________________

Student ID: ____________________

Instructions:

Please print your name, student ID.

During the test, you may not use books, calculators or telephones.

Read each question carefully, and show all your work. Disorganized writing or answers with no explanation will receive no credit, even if they are correct. *Please write your proofs neatly.*

There are 5 questions which are worth 10 points each. You have 80 minutes to complete the test.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
Extra credit		7
Total		50

Problem 1. [*10 points; 5, 5.*]

(i) Find the inverse of 11 in $\mathbb{Z}_{37}$.

(ii) Show that

$$a^{40} \equiv 1 \mod 451$$

whenever $\gcd(a, 451) = 1$. Note $451 = 11 \cdot 41$.

Problem 2. [*10 points; 3, 3, 4.*]

Consider the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \\ 5 & 4 & 2 & 3 & 6 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \\ 6 & 3 & 1 & 4 & 5 & 2 \end{pmatrix}.$$

(i) Find the permutation χ such that

$$\sigma\chi = \tau.$$

(ii) Determine the parity of σ and τ.

(iii) Using (ii), show that there are no permutations μ such that $\sigma^5 = \mu^2\tau$.

Problem 3. [*10 points; 3, 4, 3.*]

(i) Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \\ 7 & 9 & 10 & 1 & 8 & 5 & 3 & 6 & 2 & 4 \end{pmatrix}.$$

Write σ as product of three disjoint cycles $\gamma_1,\ \gamma_2,\ \gamma_3$.

(ii) Compute $\gamma_1^{60},\ \gamma_2^{60},\ \gamma_3^{60}$.

(iii) Compute σ^{60}.

Problem 4. [*10 points; 4, 6.*]

(i) If χ and τ are two permutations in S_n, show that the inverse of the permutation $\chi\tau$ is the permutation $\tau^{-1}\chi^{-1}$. That is,

$$(\chi\tau)^{-1} = \tau^{-1}\chi^{-1}.$$

(ii) On the set S_n of permutations define $\sigma_1 \sim \sigma_2$ if there exists a permutation τ such that

$$\sigma_1 = \tau \sigma_2 \tau^{-1}.$$

Show that $\sim$ defines an equivalence relation on the set S_n of permutations.

Problem 5. [*10 points; 5, 5.*]

(i) Let $p > 2$ be a prime. Prove Wilson's theorem stating that

$$(p-1)! \equiv -1 \mod p.$$

Hint: Pair every term in the product with its inverse.

(ii) Let n be a positive integer, and let π denote the product of all invertible elements in $\mathbb{Z}_n$. Show that

$$\pi^2 \equiv 1 \mod n.$$

Extra credit. [*7 points.*]

Show that there are infinitely many primes p of the form $4k+1$ for $k \in \mathbb{Z}$.

Hint: Consider $A = (2p_1 \cdots p_n)^2 + 1$.