FINAL EXAM
MATH 100A, UCSD, AUTUMN 16

You have three hours.

There are 8 problems, and the total number of points is 90. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Problem	Points	Score
1	15	
2	10	
3	15	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
Total	90	

1. (15pts) *Give the definition of a normal subgroup.*

Let G be a group and let H be a subgroup. We say that H is normal in G, if for every $g \in G$,

$$gHg^{-1} \subset H.$$

(ii) *Give the definition of a homomorphism.*

A function

$$\phi \colon G \longrightarrow G',$$

between two groups is said to be a homomorphism if for every x and $y \in G$,

$$\phi(xy) = \phi(x)\phi(y).$$

(iii) *Give the definition of A_n, the alternating group.*

The alternating group A_n is the subgroup of S_n consisting of all even permutations.

2. (10pts) *Compute* $(29)^{25} \mod 11$.

First note that $29 = 8 = -3 \mod 11$. As 11 is prime, by Fermat we have

$$a^{11} = a \mod 11.$$

Thus

$$\begin{aligned}
(29)^{25} &= (-3)^{22}(-3)^3 \\
&= -1(3)^{22}(3)^3 \\
&= -1(3)^5 \\
&= -1 \cdot 3^2 \cdot 3^2 \cdot 3 \\
&= -1 \cdot -2 \cdot -2 \cdot 3 \\
&= -12 \\
&= 10 \mod 11.
\end{aligned}$$

3. (15pts) (i) *Exhibit a proper normal subgroup V of A_4. To which group is V isomorphic to?*

$$V = \{(e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}.$$

V is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) *Give the left cosets of V inside A_4.*

$$\begin{aligned}
[e] &= V \\
[(1,2,3)] &= \{(1,2,3), (1,3,4), (2,4,3), (1,4,2))\} \\
[(1,3,2)] &= \{(1,3,2), (2,3,4), (1,2,4), (1,4,3)\}
\end{aligned}$$

(iii) *To which group is A_4/V isomorphic to?*

As it has order 3, it must be isomorphic to $\mathbb{Z}_3$.

4. (10pts) *Let* $\phi\colon G \longrightarrow G'$ *be a group homomorphism. Show that* ϕ *is injective if and only if the kernel of* ϕ *is the trivial subgroup of* G*.*

One direction is clear; if ϕ is injective then the inverse of the identity consists only of the identity, so that the kernel is the trivial subgroup. Now suppose that the kernel is the trivial subgroup. Suppose that a and $b \in G$ and $\phi(a) = \phi(b)$. Then

$$\begin{aligned}\phi(ab^{-1}) &= \phi(a)\phi(b^{-1}) \\ &= \phi(a)\phi(b)^{-1} \\ &= \phi(a)\phi(a)^{-1} \\ &= e'.\end{aligned}$$

Thus $ab^{-1} \in \operatorname{Ker}\phi$. As the kernel is the trivial subgroup, it follows that $ab^{-1} = e$. But then $a = b$. As a and b were arbitrary, it follows that ϕ is injective.

5. (10pts) *Let G be a group and let H be a subgroup. Prove that the following are equivalent.*

(1) H is normal in G.
(2) For every $g \in G$, $gHg^{-1} = H$.
(3) For every $a \in G$, $aH = Ha$.
(4) The set of left cosets is equal to the set of right cosets.

Suppose that H is normal in G. Then for all $a \in G$,

$$aHa^{-1} \subset H.$$

Taking $a = g$ and $a = g^{-1}$ we get

$$gHg^{-1} \subset H \qquad \text{and} \qquad g^{-1}Hg \subset H.$$

Multiplying the second inclusion on the left by g and on the right by g^{-1} we get,

$$H \subset gHg^{-1}.$$

Hence (2) holds. Now suppose that (2) holds. Multiplying

$$aHa^{-1} = H,$$

on the right by a, we get

$$aH = Ha.$$

Hence (3) holds. Now suppose that (3) holds. Then (4) certainly holds. Finally suppose (4) holds. Pick $g \in G$. Then $g \in gH$ and $g \in Hg$. As the set of left cosets equals the set of right cosets, it follows that $gH = Hg$. Multiplying on the right by g^{-1} we get

$$gHg^{-1} = H.$$

As g is arbitrary, it follows that H is normal in G. Hence (1). Thus the four conditions are certainly equivalent.

6. (10pts) Let G be a group and let H be a subgroup.
(i) *If H has index two then prove that H is normal in G.*

If H has index two there are only two left cosets. One of them is H and the other has to be the complement. As the number of right cosets is equal to the number of left cosets, there are only two right cosets. One of them is H and so the other one is again the complement. Thus the set of left cosets and right cosets is the same and H is normal in G.

(ii) *Give an example where H has index three and H is not normal.*

Let $G = S_3$ and $H = \{e, (1,2)\}$. Then the index of H in G is 3 by Lagrange. If we take $\sigma = (1,2) \in H$ and $\tau = (2,3) \in G$ then

$$\tau\sigma\tau^{-1} = (1,3) \notin H.$$

Thus H is not normal.

7. (10pts) *Let G be a group and let N be a normal subgroup. Prove that G/N is abelian if and only if N contains the commutator of every pair of elements of G.*

Suppose that G/N is abelian. Let a and $b \in G$ and set $x = aN$ and $y = bN$. As G/N is abelian, $xy = yx$, so that $abN = baN$ and $ab = ban$, for some $n \in N$. But then

$$a^{-1}b^{-1}ab = n \in N,$$

so that N contains the commutator of a and b.
Now suppose that N contains the commutator of any pair of elements of G. Pick x and $y \in G/N$. Then $x = aN$ and $y = bN$. We have

$$\begin{aligned} yx &= (bN)(aN) \\ &= baN \\ &= ba(a^{-1}b^{-1}ab)N \\ &= abN \\ &= xy. \end{aligned}$$

Thus G/N is abelian

8. (10pts) State and prove one of the Isomorphism Theorems.

Bonus Challenge Problems

9. (10pts) Prove the rest of the Isomorphism Theorems.

10. (10pts) Prove that there is no permutation that is both even and odd.

11. (10pts) Classify all groups of order at most ten.

12. (10pts) Show that if G is a simple group whose order is less than 60, then G has order a prime.

13. (10pts) Prove that A_n is simple, for $n \geq 5$.