Math 100A - Fall 2019 - Practice Problems for Midterm II

The midterm will cover Chapters 2.1 - 2.5 in the book. The main topics are:

- *binary laws, groups, basic properties, examples*
- *subgroups, center, centralizer, normal subgroups*
- *order of elements, properties of order, order of permutations, cyclic groups, subgroups of cyclic group*
- *homomorphisms, isomorphisms, examples, basic properties, kernel, image*
- *automorphisms, inner automorphisms*

1. Please make sure to review the definitions and all proofs covered in class. You may be asked to define terms or prove a statement which is similar to theorems proved in class.

Also please review the homework problems.

2. (*Binary laws, groups, isomorphisms.*) Consider $G = \mathbb{Q} \setminus \{-1\}$ endowed with the binary law

$$a \star b = ab + a + b.$$

(i) Show that $\star$ is a well-defined binary law on G.
(ii) Show that $(G, \star)$ is a group with 0 as identity.
(iii) Show that $(G, \star)$ is isomorphic to $(\mathbb{Q} \setminus \{0\}, \cdot)$

3. (*Groups.*) Show that if G is a group with an even number of elements, there exists $a \in G$, $a \neq e$ such that $a^2 = e$. You may wish to group elements in pairs (a, a^{-1}).

4. (*Subgroups, normal subgroups.*) Let H be a subgroup of a group G. The normalizer of H in G is defined as

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

(i) Show that $N_G(H)$ is a subgroup of G.
(ii) Show that H is normal if and only if $N_G(H) = G$.

5. (*Subgroups.*) If X is a subset of a group G, define

$$\langle X \rangle = \{x_1^{k_1} \cdots x_m^{k_m} : \ x_i \in X, \ k_i \in \mathbb{Z}, \ m \geq 1\}.$$

(i) Show that $\langle X \rangle$ is a subgroup of G.
(ii) Show that if $G = \langle X \rangle$ and $xy = yx$ for all $x, y \in X$ then G is abelian.

6. (*Normal subgroups.*) Show that $H = \{1, \sigma, \sigma^2\}$ is a normal subgroup of $G = S_3$. There are two ways of solving this problem. One involves direct verification (which is tedious and I won't recommend).

A second method is based on a general argument. If $g \in G \backslash H$, show that H and gH are disjoint. Similarly show that H and Hg are disjoint. Comparing the number of elements of G, H, gH and Hg conclude that $Hg = gH$ so that H is normal.

7. (*Order of permutations.*) Find the smallest integer n such that $\sigma^n = \epsilon$ for all $\sigma \in S_6$.

8. (*Cyclic groups and their subgroups.*) Let $G = \langle g \rangle$ be a finite cyclic group of order n. Show that if $a|n$ and $b|n$ then

$$\langle g^a \rangle \cap \langle g^b \rangle = \langle g^c \rangle$$

where c is the least common multiple of a, b.

9. (*Order of elements.*) Let $g \in G$ be an element of order m, and let $h \in H$ be an element of order n. Find the order of the element (g, h) in $G \times H$.

10. (*Automorphisms.*) If G is an infinite cyclic group, find $\mathrm{Aut}(G)$.

11. (*Isomorphisms. Order.*) Exhibit, with proof, three nonisomorphic groups of order 27.

12. (*Cyclic groups, their subgroups. Automorphisms.*)

(i) For $n = p_1^{a_1} \cdots p_k^{a_k}$ with p_i prime, find the number of subgroups of C_n.
(ii) Draw the lattice of subgroups of $C_{p^2q^2}$, where p, q are prime.
(iii) Find the number of automorphisms of C_{12}.

13. (*Inner automorphisms.*) Show that if $Z(G) = \{1\}$, then $G \simeq \mathrm{Inn}(G)$.