1. (a) By the definition of $\ell(G)$, it is clear that

$$\ell(G') \leq \ell(G) - 1$$

where G' is the game after the first player makes her move.

Now by strong induction on $\ell(G_1)+\ell(G_2)$, we prove that $\ell(G_1 \oplus G_2) \leq \ell(G_1) + \ell(G_2)$.

<u>Base</u> $\ell(G_1) + \ell(G_2) = 0$

In this case, $\ell(G_1) = \ell(G_2) = 0$ which means the first player has no move to make in either of the games. So he has no move to make in $G_1 \oplus G_2$, which implies $\ell(G_1 \oplus G_2) = 0$.

<u>Inductive step</u>. $\left(\ell(G_1)+\ell(G_2) \leq k \Rightarrow \ell(G_1 \oplus G_2) \leq \ell(G_1) + \ell(G_2)\right)$

$$\Downarrow ?$$

$$\left(\ell(G_1) + \ell(G_2) = k+1 \Rightarrow \ell(G_1 \oplus G_2) \leq \ell(G_1) + \ell(G_2)\right)$$

<u>Pf</u>. The first player makes her move in one and only one of the games G_1 or G_2. If she plays in G_1, then G_1 changes to a game G_1' and $G_1 \oplus G_2$ changes to

$G_1' \oplus G_2$. Since $\ell(G_1') + \ell(G_2) \leq \ell(G_1) - 1 + \ell(G_2)$ $= (k+1) - 1 = k$, by the induction hypothesis we have $\ell(G_1' \oplus G_2) \leq \ell(G_1') + \ell(G_2) = \ell(G_1) + \ell(G_2) - 1$.

This shows that, if the first player plays in G_1, then $G_1 \oplus G_2$ finishes in at most $\underbrace{1}_{\text{move of the first player}} + \ell(G_1' \oplus G_2) \leq \ell(G_1) + \ell(G_2)$

A similar argument shows that if the first player plays in G_2, then again $G_1 \oplus G_2$ finishes in at most $\ell(G_1) + \ell(G_2)$. This proves that $G_1 \oplus G_2$ is finite and $\ell(G_1 \oplus G_2) \leq \ell(G_1) + \ell(G_2)$.

(b) The idea is simple: The second player copies the first player till the first player is out of move, i.e. loses!

Formal proof. We proceed by strong induction on $\ell(G)$.

Base $\ell(G) = 0 \implies$ The first player has no move to make in G so he has no move to make in $G \oplus G$. Hence The first player loses.

<u>The inductive step</u>. $\ell(G) \leq k \Rightarrow G \oplus G$ is N.

$$\Downarrow ?$$

$$\ell(G) = k+1 \Rightarrow G \oplus G \text{ is } N.$$

<u>Pf</u>. We have to show for any move of the first player the second player has a move to turn the game to an N (a losing game for the first player).

Assume the first player makes a move $\underline{\underline{M}}$ in one of the copies of G and change the game to G'. This changes $G \oplus G$ to $G' \oplus G$.

Now the second player can make the same move $\underline{\underline{M}}$ in the other copy of G. This changes the game to $G' \oplus G'$.

Since $\ell(G') \leq \ell(G) - 1 = k$, by induction hypothesis $G' \oplus G'$ is N, as we desired.

2. We proceed by strong induction on $\ell(G_1)+\ell(G_2)$.

<u>Base</u> $\ell(G_1)+\ell(G_2)=0 \Rightarrow \ell(G_1)=\ell(G_2)=0$

$\Rightarrow$ the first player has no move to make in neither G_1 nor $G_2 \Rightarrow G_1$ and G_2 and $G_1 \oplus G_2$ are N.

<u>Inductive step.</u>

The strong induction hypothesis

$$\boxed{\begin{array}{l} \ell(G_1)+\ell(G_2)\le k \\ G_1 : P \wedge G_2 : N \Rightarrow G_1 \oplus G_2 : P \\ G_1 : N \wedge G_2 : N \Rightarrow G_1 \oplus G_2 : N \end{array}}$$

$$\Downarrow \ ?$$

$$\boxed{\begin{array}{l} \ell(G_1)+\ell(G_2)= k+1 \\ G_1 : P \wedge G_2 : N \Rightarrow G_1 \oplus G_2 : P \\ G_1 : N \wedge G_2 : N \Rightarrow G_1 \oplus G_2 : N \end{array}}$$

First assume $G_1 : P \wedge G_2 : N$, we would like to show that player A has a move which makes it an N-game.

Since G_1 is a P-game, the first player has a move which makes it G_1' which is an N-game. After this move

$G_1 \oplus G_2$ changes to $G_1' \oplus G_2$.

$$\left.\begin{array}{l} G_1' : N \wedge G_2 : N \\ \ell(G_1') + \ell(G_2) \le \ell(G_1) + \ell(G_2) - 1 \end{array}\right\} \xrightarrow[\text{hypothesis}]{\text{by induction}} G_1' \oplus G_2 \text{ is an } N\text{-game as we wished.}$$

Assume $G_1 : N$ and $G_2 : N$. We would like to prove that no matter what player A does, the game will be changed to a winning game for player B, i.e. a P-game (notice that after player A's move, player B is the first player of the new game.)

Player A has make a move in one and only one of G_1 and G_2. Because of the symmetry, we can and will assume that A makes his move in G_1 and it gives us a new game G_1'. Since G_1 is an N-game, G_1' is definitely a P-game. So $G_1 \oplus G_2$ changes to $G_1' \oplus G_2$.

$$\left.\begin{array}{l} \ell(G_1') + \ell(G_2) \le k \\ G_1' : P \wedge G_2 : N \end{array}\right\} \xrightarrow[\text{hypothesis}]{\text{The induction}} G_1' \oplus G_2 \text{ is a } P\text{-game as we wished.} \blacksquare$$

3. Construct a proof backwards

$$|x^2-9| \le \varepsilon \Longleftarrow |x-3||x+3| \le \varepsilon \Longleftarrow |x-3| \le \varepsilon/7 \wedge |x+3| \le 7 .$$

The idea behind this inequality

if $|x-3| \le 1$, then

$$2 \le x \le 4 \Longrightarrow 5 \le x+3 \le 7$$
$$\Longrightarrow |x+3| \le 7$$

$$\Longleftarrow |x-3| \le \varepsilon/7 \wedge |x-3| \le 1$$

The above argument show

$$\forall \varepsilon > 0, \quad |x-3| \le \min\{1, \varepsilon/7\} \Longrightarrow |x^2-9| \le \varepsilon .$$

is 1 if $\varepsilon \ge 7$ and $\varepsilon/7$ if $\varepsilon < 7$

This implies

$$\forall \varepsilon > 0, \exists \delta > 0, \quad |x-3| \le \delta \Longrightarrow |x^2-9| \le \varepsilon . \qquad \blacksquare$$

4. Suppose to the contrary that

$$\neg \left[\forall n \in \mathbb{Z}^{>1}, \left(\left(\nexists m \in \mathbb{Z}^{>1}, \ m \le \sqrt{n} \wedge m \mid n \right) \Longrightarrow n \text{ is prime} \right) \right]$$

holds. Let's see what it means

$$\exists n \in \mathbb{Z}^{>1}, \ \neg \left(\left(\nexists m \in \mathbb{Z}^{>1}, \ m \le \sqrt{n} \wedge m \mid n \right) \Longrightarrow n \text{ is prime} \right)$$

which is equivalent to

$\exists n \in \mathbb{Z}^{>1}, \quad (\nexists m \in \mathbb{Z}^{>1},\ m \le \sqrt{n} \wedge m | n) \wedge$ (n is NOT prime).

If $n > 1$ and n is NOT a prime, then by the definition

$$\exists d \in \mathbb{Z}, \quad d | n \wedge 1 < d < n$$

Thus $n = d \cdot \frac{n}{d}$, $\quad d, \frac{n}{d}$ are integers and

$$1 < d, \frac{n}{d} < n. \quad \text{I}$$

<u>Claim</u> In the above setting, either $d \le \sqrt{n}$ or $\frac{n}{d} \le \sqrt{n}$. II

<u>Pf.</u> If not, $\quad d > \sqrt{n}$ and $\frac{n}{d} > \sqrt{n}$ which implies

$$n = d \cdot \frac{n}{d} > \sqrt{n} \cdot \sqrt{n} = n$$

that is a contradiction.

Hence by I, II we have that

If $n \in \mathbb{Z}^{>1}$ is NOT a prime, then

$$\exists d' \in \mathbb{Z}, \ 1 < d' \le \sqrt{n} \wedge d' | n.$$

The above result contradicts our assumptions on $\underline{\underline{n}}$; namely

$(\nexists m \in \mathbb{Z}^{>1},\ m \le \sqrt{n} \wedge m | n) \wedge (n$ is NOT prime$)$. ■

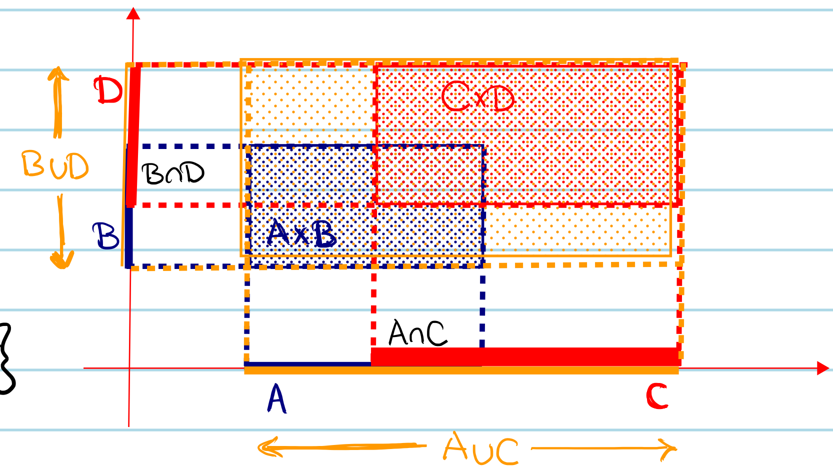
5. (a) False. Let $A = \{1, 2\}$, $B = \{a, b\}$, $C = \{2, 3\}$

and $D = \{b, c\}$.

Then

$A \cup C = \{1, 2, 3\}$

and $B \cup D = \{a, b, c\}$



So $(3, a) \in (A \cup C) \times (B \cup D)$, but

$(3,a) \notin A \times B$ as $3 \notin A$ and $(3,a) \notin C \times D$ as $a \notin D$. $\blacksquare$

(b) $(x,y) \in (A \times B) \cup (C \times D) \Rightarrow (x,y) \in A \times B \vee (x,y) \in C \times D$

$$\Rightarrow (x \in A \wedge y \in B) \vee (x \in C \wedge y \in D)$$

$$\Rightarrow (x \in A \cup C \wedge y \in B \cup D) \vee (x \in A \cup C \wedge y \in B \cup D)$$

$$\Rightarrow x \in A \cup C \wedge y \in B \cup D$$

$$\Rightarrow (x,y) \in (A \cup C) \times (B \cup D).$$

So $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. $\blacksquare$

Alternatively

$$\left.\begin{array}{l}\left.\begin{array}{l} A \subseteq A \cup C \\ B \subseteq B \cup D \end{array}\right\} \Rightarrow A \times B \subseteq (A \cup C) \times (B \cup D) \\ \left.\begin{array}{l} C \subseteq A \cup C \\ D \subseteq B \cup D \end{array}\right\} \Rightarrow C \times D \subseteq (A \cup C) \times (B \cup D) \end{array}\right\} \Rightarrow$$

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D). \blacksquare$$

(c) $(x,y) \in (A \times B) \cap (C \times D) \iff (x,y) \in A \times B \wedge (x,y) \in C \times D$

$\iff x \in A \wedge y \in B \wedge x \in C \wedge y \in D$

$\iff (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D)$

$\iff x \in A \cap C \wedge y \in B \cap D$

$\iff (x,y) \in (A \cap C) \times (B \cap D).$

$\Rightarrow (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$ ∎

6.(a) This is incomplete as the domains and codomains of g_i's are NOT given.

$$\mathbb{R} \xrightarrow{g_1} ? \xrightarrow{g_2} ? \xrightarrow{g_3} \mathbb{R}$$

$$x \longmapsto x^2 \longmapsto x^2+1 \longmapsto \sqrt{x^2+1}$$

The only restriction upon us is the fact that $g_3(x) = \sqrt{x}$ is NOT defined at negative numbers. So let $g_3 : \mathbb{R}^{\geq 0} \to \mathbb{R}$. In order to be able to talk about the composite function $\underset{\sim\sim}{g_3 \circ g_2}$ the codomain of g_2 should be the domain of g_3. So $g_2 : ? \to \mathbb{R}^{\geq 0}$.

This implies for any x in the domain of g_2

we have $g_2(x) = x+1 \geq 0$ so $x \geq -1$. Hence let $g_2 : \mathbb{R}^{\geq -1} \longrightarrow \mathbb{R}^{\geq 0}$. Now we notice that

$$g_1 : \mathbb{R} \to \mathbb{R}^{\geq -1}, \quad g_1(x) = x^2$$

is well-defined as, for any $x \in \mathbb{R}$, $x^2 \geq 0 \geq -1$.

Summary Let $g_1 : \mathbb{R} \to \mathbb{R}^{\geq -1}$, $g_1(x) = x^2$;

$$g_2 : \mathbb{R}^{\geq -1} \longrightarrow \mathbb{R}^{\geq 0}, \quad g_2(x) = x+1;$$

$$g_3 : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}, \quad g_3(x) = \sqrt{x}$$

Then $f = g_3 \circ g_2 \circ g_1 : \mathbb{R} \longrightarrow \mathbb{R}$ s.t.

$$f(x) = \sqrt{x^2+1}. \qquad \blacksquare$$

(b) This is NOT true. In fact, since the domain of is NOT equal to its codomain, $f \circ f$ is NOT defined.

We notice that $\frac{1}{x} \neq 0$ if $x \in \mathbb{R} \setminus \{0\}$. So we can modify f to a new function:

$$g : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\} \text{ s.t. } g(x) = \frac{1}{x}$$

[The only difference between f and g is their codomain.]

Now we can talk about $g \circ g : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$.

And clearly $g \circ g(x) = g(1/x) = 1/(\frac{1}{x}) = x$. Hence

$g \circ g = id._{\mathbb{R} \setminus \{0\}}$ where $id._{\mathbb{R} \setminus \{0\}} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$, $id(x) = x$.

[The only difference between $id._{\mathbb{R}}$ and $id._{\mathbb{R} \setminus \{0\}}$ is their domain and codomain.] ■

7. Suppose to the contrary that there is $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f(x) = x$. Hence $g \circ f(1) = 1$ and $g \circ f(-1) = -1$ which implies $g(1) = 1$ and $g(1) = -1$. That is a contradiction. ■