1. Clearly there is a bijection between the possible paths and elements of $\{1,2,3,4\} \times \{a,b,c,d,e\}$. So by the multiplication principle, the number of possible paths is equal to $(4)(5)=20$.

2. We prove that $|A_1 \times A_2 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_k|$ if $|A_i| < \infty$ by induction on k.

<u>Base</u> $k=1$ is clear.

<u>Inductive step</u>. $|A_1 \times \cdots \times A_k| = \prod_{i=1}^{k} |A_i| \Rightarrow |A_1 \times \cdots \times A_{k+1}| = \prod_{i=1}^{k+1} |A_i|$.

<u>Proof of the inductive step</u>.

<u>Claim</u>. $f: (A_1 \times \cdots \times A_k) \times A_{k+1} \longrightarrow A_1 \times \cdots \times A_{k+1}$

$$f((a_1, \cdots, a_k), a_{k+1}) = (a_1, \cdots, a_{k+1})$$

is a bijection.

<u>Pf of the claim</u>. <u>f is 1-1</u>.

$$f((a_1, \cdots, a_k), a_{k+1}) = f((a_1', \cdots, a_k'), a_{k+1}')$$

$$\Rightarrow (a_1, \cdots, a_{k+1}) = (a_1', \cdots, a_{k+1}')$$

$$\Rightarrow a_1 = a_1', \; a_2 = a_2', \; \cdots, a_{k+1} = a_{k+1}'$$

$$\Rightarrow ((a_1, \cdots, a_k), a_{k+1}) = ((a_1', \cdots, a_k'), a_{k+1}')$$

- f is onto.

For any $(a_1, \ldots, a_{k+1})$ we have $f((a_1, \cdots, a_k), a_{k+1}) = (a_1, \ldots, a_{k+1})$ ▦

Hence by the above claim we have

$$|A_1 \times \cdots \times A_{k+1}| = |(A_1 \times \cdots \times A_k) \times A_{k+1}|$$

$$= |A_1 \times \cdots \times A_k| \cdot |A_{k+1}| \quad \text{(by the mult. principle.)}$$

$$= (|A_1| \cdot |A_2| \cdot \cdots \cdot |A_k|) \cdot |A_{k+1}| \quad \text{(induction hypothesis)}$$

$$= \prod_{i=1}^{k+1} |A_i|.$$

3, 4. The main idea behind the solutions of 3 and 4:

Claim There is a bijection between the set

$$\text{Fun}(\{1, 2, \ldots, n\}, B)$$

of functions from $\{1, 2, \ldots, n\}$ and

$$\underbrace{B \times B \times \cdots \times B}_{n \text{ copies}}.$$

Pf of the claim.

Let $g : \text{Fun}(\{1, 2, \cdots, n\}, B) \longrightarrow B \times B \times \cdots \times B$,

$$g(f) := (f(1), f(2), \cdots, f(n)).$$

5. We use inclusion-exclusion:

$$|X \setminus (A\cup B\cup C)| = |X| - |A| - |B| - |C| + |A\cap B| + |A\cap C| + |B\cap C| - |A\cap B\cap C|.$$

The following is the key fact:

Claim. For any positive integers d and n, if $d \mid n$, then

$$|\{k\in\mathbb{Z} \mid 1\le k\le n,\ d\mid k\}| = n/d.$$

Pf of claim We show that

$$f: \{1,2,\dots,n/d\} \longrightarrow \{k\in\mathbb{Z} \mid 1\le k\le n, d\mid k\}$$

$$f(i) = di$$

is a bijection.

① f is well-defined.

- $1\le i\le n/d \Rightarrow d\le di\le n \Rightarrow 1\le di\le n$
- $d \mid di$.

② f is 1-1.

$$f(i_1)=f(i_2) \Rightarrow di_1 = di_2 \Rightarrow i_1 = i_2.$$

③ f is onto.

$d\mid k \Rightarrow k = di$ for some $i\in\mathbb{Z}$

$$1 \le k \le n \Rightarrow 1 \le di \le n$$

$$\left.\begin{array}{l} \Rightarrow \frac{1}{d} \le i \le \frac{n}{d} \\ i \in \mathbb{Z},\ \frac{1}{d} > 0 \end{array}\right\} \Rightarrow 1 \le i \le \frac{n}{d}$$

$$\Rightarrow k = f(i) \text{ for some } i \in \{1, 2, \dots, \tfrac{n}{d}\}. \blacksquare$$

Using the above claim: $|A| = 300/2 = 150$

$$|B| = 300/3 = 100$$

$$|C| = 300/5 = 60$$

$$A \cap B = \{k \in \mathbb{Z} \mid 1 \le k \le 300,\ 6 \mid k\}$$

$$A \cap C = \{k \in \mathbb{Z} \mid 1 \le k \le 300,\ 10 \mid k\}$$

$$B \cap C = \{k \in \mathbb{Z} \mid 1 \le k \le 300,\ 15 \mid k\}$$

$$A \cap B \cap C = \{k \in \mathbb{Z} \mid 1 \le k \le 300,\ 30 \mid k\}$$

$$\Rightarrow |A \cap B| = 300/6 = 50$$

$$|A \cap C| = 300/10 = 30$$

$$|B \cap C| = 300/15 = 20$$

$$|A \cap B \cap C| = 300/30 = 10$$

$$\Rightarrow |X \setminus (A \cup B \cup C)| = 300 - 150 - 100 - 60 + 50 + 30 + 20 - 10$$

$$= 80$$

6. By Pigeonhole principle at least two points P_i and P_j share "the same pigeonhole" a, b, c or d.

$\Rightarrow |P_i P_j| \le$ diameter of a $(1/2) \times (1/2)$ square

$$= \frac{1}{\sqrt{2}}.$$