1. (The First Solution)

ⓐ We proceed by induction on n.

Base of the induction: $8^1+6=14=2\times 7 \Rightarrow 7 \mid 8^1+6$.

The induction step: $7 \mid 8^k+6 \Rightarrow 7 \mid 8^{k+1}+6$ (?)

$$8^{k+1}+6 = 8\left[(8^k+6)-6\right]+6 \quad \text{(We write it this way to use the induction hypothesis)}$$
$$= 8[7\cdot \ell -6]+6 \quad \text{(by the induction hypothesis } 7 \mid 8^k+6.)$$
$$= 56\ell - 42$$
$$= 7(8\ell-6)$$
$$\Rightarrow 7 \mid 8^{k+1}+6.$$

ⓑ By induction on n, we prove that $7 \mid 6^n-(-1)^n$.

Base of the induction: $6^1-(-1)^1 = 7 \Rightarrow 7 \mid 6^1-(-1)^1$.

The induction step: $7 \mid 6^k-(-1)^k \Rightarrow 7 \mid 6^{k+1}-(-1)^{k+1}$. (?)

$$6^{k+1}-(-1)^{k+1} = 6\left[(6^k-(-1)^k)+(-1)^k\right]-(-1)^{k+1}$$
$$= 6\left[7\ell+(-1)^k\right]+(-1)^k \quad \text{[by the induction hypothesis]}$$
$$= 7\left[6\ell+(-1)^k\right]$$
$$\Rightarrow 7 \mid 6^{k+1}-(-1)^{k+1}.$$

If n is even, then $7 \mid 6^n-1 \Rightarrow 7 \mid (6^n-1)+7 = 6^n+6$.

If n is odd, then $7 \mid 6^n+1$. If to the contrary $7 \mid 6^n+6$, then

$$7 \mid (6^n+6)-(6^n+1)=5$$

which is a contradiction.

Hence $7 \mid 6^n+6 \iff n$ is even. ∎

(The second solution.) In the last week's homework assignment, you proved that

$$\left.\begin{matrix} a \mid b_1-c_1 \\ a \mid b_2-c_2 \end{matrix}\right\} \Rightarrow a \mid b_1b_2 - c_1c_2.$$

Using this result, by induction on n, we prove that

$$a \mid b-c \Rightarrow a \mid b^n - c^n.$$

Base of the induction. $a \mid b^1 - c^1$. ✓

The induction step. $a \mid b^k - c^k \Rightarrow a \mid b^{k+1} - c^{k+1}$ (?)

$$\left.\begin{array}{l} a \mid b^k - c^k \\ a \mid b-c \end{array}\right\} \Rightarrow a \mid b.b^k - c.c^k = b^{k+1} - c^{k+1}.$$

(The mentioned problem.)

$$\Rightarrow \begin{cases} 7 \mid 8-1 \Rightarrow 7 \mid 8^n - 1. \\ 7 \mid 6-(-1) \Rightarrow 7 \mid 6^n - (-1)^n. \end{cases}$$

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(The third solution) If $a \mid b-c$, then for any positive integer n we have

$$b^n - c^n = (b-c)(b^{n-1} + b^{n-2}c + b^{n-3}c^2 + \cdots + bc^{n-2} + c^{n-1})$$

$$= a.\ell.\left(\sum_{i=0}^{n-1} b^i c^{n-1-i}\right) \Rightarrow a \mid b^n - c^n.$$

Now we can continue as above.

■

2. Let $f(x) = \dfrac{6x+5}{x+2} = 6 - \dfrac{7}{x+2}$.

Claim 1. $f(x) > 0$ if $x > 0$.

Pf of claim 1. $x > 0 \Rightarrow x+2 > 2$ and $6x+5 > 5$

$$\Rightarrow f(x) > 0.$$

Claim 2. $f(x)$ is increasing if $x > 0$.

Pf of claim 2. $x > y > 0 \Rightarrow x+2 > y+2 > 0$

$$\Rightarrow \frac{1}{y+2} > \frac{1}{x+2}$$

$$\Rightarrow \frac{-7}{x+2} > \frac{-7}{y+2}$$

$$\Rightarrow f(x) > f(y).$$

By induction on n, we prove that $0 < a_n < 5$.

Base of induction. $0 < a_1 = 1 < 5$.

The induction step. $0 < a_k < 5 \implies 0 < a_{k+1} < 5$ (?)

We know that $a_{k+1} = f(a_k)$

$0 < a_k$ (by the induction hypothesis) $\implies 0 < f(a_k) = a_{k+1}$
(by Claim 1.)

$0 < a_k < 5$ (by the induction hypothesis) $\implies f(a_k) < f(5)$
(by Claim 2.)

$\implies a_{k+1} < 5$. ∎

3. We proceed by induction on n.

Base of the induction. $1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} = \frac{2+1}{2\times 2}$ ✓

The induction step. $\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right) = \frac{k+1}{2k} \implies \prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) = \frac{k+2}{2(k+1)}$ (?)

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) = \left[\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\right]\cdot\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k}\cdot\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$ (by the induction hypothesis)

$$= \frac{k+1}{2k}\cdot\frac{[(k+1)-1][(k+1)+1]}{(k+1)^2}$$

$$= \frac{(k+1)\cdot k\cdot(k+2)}{2k(k+1)^2}$$

$$= \frac{k+2}{2(k+1)}.$$ ∎

4. As it was discussed in class, we proceed by strong induction.

Base of the induction. $n = 2$ ✓ (as 2 is a prime.)

The strong induction step. We assume any integer $2 \le k \le n$ can be written as product of primes. We would like to prove that $n+1$ can be written as product of primes.

If $n+1$ is a prime, there is nothing to prove.

If $n+1$ is NOT a prime, then there is an integer d such that

1. $d \mid n+1$,
2. $1 < d < n+1$.

Hence $n+1 = d \cdot \left(\frac{n+1}{d}\right)$ and $2 \le d, \frac{n+1}{d} \le n$. So by the strong induction hypothesis, d and $\frac{n+1}{d}$ can be written as product of primes.

Therefore $n+1 = d \cdot \left(\frac{n+1}{d}\right)$ can be written as product of primes. ■

5. (a)

$x \in A$	$x \in B$	$x \in A \setminus B$	$x \in B \setminus A$	$x \in A \Delta B$	$x \in A \cup B$	$x \in A \cap B$	$x \in (A \cup B) \setminus (A \cap B)$
T	T	F	F	F	T	T	F
T	F	T	F	T	T	F	T
F	T	F	T	T	T	F	T
F	F	F	F	F	F	F	F

So $x \in A \Delta B \equiv x \in (A \cup B) \setminus (A \cap B)$. Hence $A \Delta B = (A \cup B) \setminus (A \cap B)$.

(b)

$x \in A$	$x \in B$	$x \in C$	$x \in A \Delta B$	$x \in (A \Delta B) \Delta C$	$x \in B \Delta C$	$x \in A \Delta (B \Delta C)$
T	T	T	F	T	F	T
T	T	F	F	F	T	F
T	F	T	T	F	T	F
T	F	F	T	T	F	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

So $x \in A \triangle (B \triangle C) \equiv x \in (A \triangle B) \triangle C$. Thus $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

[To write the above table, we used 5.a's truth-table; namely we know $x \in S \triangle T$ is true if and only if one and exactly one of the following is true ① $x \in S$ ② $x \in T$.]

ⓒ $A \triangle A = (A \setminus A) \cup (A \setminus A) = \emptyset \cup \emptyset = \emptyset$.

ⓓ $A \triangle \emptyset = (A \setminus \emptyset) \cup (\emptyset \setminus A) = A \cup \emptyset = A$. ∎

6. $A \triangle C = B \Rightarrow A \triangle (A \triangle C) = A \triangle B$

$\Rightarrow (A \triangle A) \triangle C = A \triangle B$ (By 5.b)

$\Rightarrow \emptyset \triangle C = A \triangle B$ (By 5.c)

$\Rightarrow C = A \triangle B$ (By 5.d)

So if there is such subset, then it is unique and it has to be $A \triangle B$.

$A \triangle (A \triangle B) = (A \triangle A) \triangle B = \emptyset \triangle B = B$.

This shows that $A \triangle B$ is such subset. ∎