

# More on Fundamental Regions

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## 1 Geometric Finiteness

**Definition** *Geometric Finiteness* If there is a convex fundamental region for a Fuchsian group  $\Gamma$  with finitely many sides, we say  $\Gamma$  is *geometrically finite*.

**Theorem 1.1** Siegel's Theorem *If  $\mu(\Gamma \backslash \mathbb{H}) < \infty$  for a Fuchsian group  $\Gamma$ , then  $\Gamma$  is geometrically finite.*

**Proof** If  $F$  is compact, then it must have finitely many edges, because each edge of  $D_p(\Gamma)$  is a perpendicular bisector of the point  $p$  and some element of  $\Gamma p$ , and  $\Gamma p$  is an isolated set. This means we can easily triangulate  $F$  and calculate its finite area using the Gauss-Bonnet theorem. So we restrict our attention to situations where  $F$  is not compact and therefore has a vertex on  $\mathbb{R} \cup \{\infty\}$ .

Let  $F = D_p(\Gamma)$  be any Dirichlet region. Connect each vertex on the boundary  $\partial F$  to the point  $p$  with geodesics to get a triangulation of  $F$ . Next, pick a set of segments on  $\partial F$  that are connected and label them  $A_m, A_{m+1}, \dots, A_n$ . Denote the vertex on one side of  $A_m$  by  $a_m$  so that the previous set of segments have vertices  $a_m, \dots, a_{n+1}$ . Let  $\Delta_k$  be the triangle having side  $A_k$  and denote its angles  $\alpha_k, \beta_k$ , and  $\gamma_k$  where if  $\omega_k$  is the angle at vertex  $a_k$ , then  $\omega_{k+1} = \beta_k + \gamma_{k+1}$ . From the Gauss-Bonnet theorem,  $\mu(\Delta_k) = \pi - \alpha_k - \beta_k - \gamma_k$ . It follows that

$$\sum_{k=m}^{n-1} \mu(\Delta_k) = (n-m)\pi + (\beta_{m-1} + \gamma_n - \sum_{k=m}^n \omega_k) - \sum_{k=m}^{n-1} \alpha_k$$

or equivalently,

$$\sum_{k=m}^{n-1} \mu(\Delta_k) + \sum_{k=m}^{n-1} \alpha_k = -\pi + \beta_{m-1} + \gamma_n + \sum_{k=m}^n \pi - \omega_k$$

Now, hold  $m$  fixed and let  $n \rightarrow \infty$ . Examining the right side, we see that the first sum is bounded since  $\mu(\Gamma \backslash \mathbb{H}) < \infty$  and the second is bounded by  $2\pi$ . Therefore, the right side must also be bounded. As a result,  $\sum_{k=m}^n \pi - \omega_k$ , which is strictly increasing must converge and  $\gamma_n$  has a limit, which we shall call  $\gamma_\infty$ . Since the sum on the right side converges, we see that  $\pi - \omega_k = \pi - \gamma_k - \beta_{k-1} \rightarrow 0^+$ . So there is also a limit  $\beta_\infty$  and  $\beta_\infty + \gamma_\infty = \pi$ . Furthermore, since  $\rho(p, a_k)$  must be unbounded, for infinitely many  $k$ ,  $\rho(p, a_{k+1}) > \rho(p, a_k)$ , which means by the sine rule  $\left(1 < \frac{\sinh \rho(p, a_{k+1})}{\sinh \rho(p, a_k)} = \frac{\sin \gamma_k}{\sin \beta_k}\right)$  that  $\gamma_k > \beta_k$ , so  $\gamma_\infty \geq \beta_\infty$ , which means

$\gamma_\infty \geq \pi/2$ . Likewise there is a  $\beta_{-\infty} \geq \pi/2$ . So  $-\pi + \beta_{-\infty} + \gamma_\infty \geq 0$ , which means

$$\sum_{k=-\infty}^{\infty} \alpha_k + \sum_{k=-\infty}^{\infty} \mu(\Delta_k) \geq \sum_{k=-\infty}^{\infty} (\pi - \omega_k)$$

And adding the previous inequality for all connected parts of  $\partial F$ , we get  $2\pi + \mu(F) \geq \sum_{\omega} \pi - \omega$  where  $\omega$  belongs to any vertex with finite hyperbolic distance from  $p$ . We wish to show that there are only finitely many such angles.

Let  $a^{(1)}, \dots, a^{(n)}$  be vertices with finite hyperbolic distance from  $p$  that are congruent modulo  $\Gamma$ . Say  $a^{(i)}$  has angle  $\omega^{(i)}$ . Then if  $m$  is the order of the element that fixes  $a^{(1)}$ , then  $\sum_i \omega^{(i)} = 2\pi/m$ . We only include vertices  $\omega^i < \pi$ , so  $n \geq 3$ , which means  $\sum_{i=1}^n (\pi - \omega^{(i)}) = (n - \frac{2}{m})\pi \geq \pi$ . Therefore, there are only a finite number of sets of congruent vertices and each set contains a finite number of vertices.

The last step is to show that the remaining vertices (at infinity) have finite cardinality. We can select any  $N$  of them and make a hyperbolic polygon  $F_1 \subset F$  having the same  $N$  vertices at infinity and get an analogous equation  $\sum_{\omega} (\pi - \omega) = 2\pi + \mu(F_1)$ . Since  $\omega = 0$  for any vertex at infinity, the sum is no less than  $N\pi$ . Thus,  $N\pi \leq 2\pi + \mu(F_1) \leq 2\pi + \mu(F)$ , which means  $N$  is bounded. ■

**Theorem 1.2** *If  $\Gamma \backslash \mathbb{H}$  is compact for a Fuchsian group  $\Gamma$ , then there are no parabolic elements of  $\Gamma$ .*

**Proof** Denote by  $F$  a compact Dirichlet region for  $\Gamma$  and define a function  $\eta$  on all  $z \in \mathbb{H}$  by the minimum distance between  $z$  and any one of its images under a non-elliptic element of  $\Gamma$ .  $\eta(z) = \inf\{\rho(z, T(z)) | T \in \Gamma, T \text{ not elliptic}\}$ . First, we should show that  $\eta(z)$  is continuous. Note that since  $\Gamma z_0$  is discrete, only a finite number of non-elliptic elements even approach the infimum, so we can ignore the others. Given any small  $\epsilon > 0$ , let  $\delta = \epsilon/3$ . If  $\rho(z, z_0) < \delta$ , then for any non-elliptic  $T_i \in \Gamma$ ,  $\rho(T_i(z), T_i(z_0)) < \delta$ . This means  $\rho(z, T_i(z)) \leq \rho(z_0, T_i(z_0)) + 2\delta \leq \eta(z_0) + 2\delta$ . So  $|\eta(z) - \eta(z_0)| \leq 2\delta < \epsilon$ , which means  $\eta(z)$  is continuous.

Since  $F$  is compact, there is some  $z_0 \in F$  so that  $\forall z \in F, \eta(z_0) \leq \eta(z)$ . Denote  $\eta(z_0) = \eta$ . Note, that since we are excluding elliptic points,  $z_0$  is not fixed and  $\eta > 0$ . In fact, this inequality holds throughout  $\mathbb{H}$ . Let  $z \in \mathbb{H}$  be an arbitrary point and pick  $w \in F$  and  $S \in \Gamma$  such that  $w = S(z)$ . Then  $\rho(z, T_0(z)) = \rho(S(z), ST_0(z)) = \rho(w, ST_i S^{-1}(w)) \geq \eta$ .

If  $\Gamma$  were to contain an elliptic element, then it would be conjugate in  $PSL(2, \mathbb{R})$  to the transformation  $z \mapsto z + 1$ . But picking some  $z_0$  with very large imaginary value, we get  $\rho(z_0, z_0 + 1) < \eta$ , which is a contradiction. ■

## 2 Fuchsian Group Signatures

For this section, let  $\Gamma$  have a compact fundamental region  $F$ . Since  $F$  has finitely many vertices,  $\Gamma$  contains finitely many elliptic cycles and hence a finite number of periods, given by  $m_1, \dots, m_r$ . The quotient space  $\Gamma \backslash \mathbb{H}$  is a compact orientable surface, called an orbifold, with genus  $g$ . Given this information, we say that  $\Gamma$  has signature  $(g; m_1, \dots, m_r)$ .

**Theorem 2.1** *If a Fuchsian group  $\Gamma$  has signature  $(g; m_1, m_2, \dots, m_r)$ , then*

$$\mu(\Gamma \backslash \mathbb{H}) = 2\pi \left[ (2g - 2) + \sum_{i=1}^r \left( 1 - \frac{1}{m_i} \right) \right]$$

**Proof** The Dirichlet region has  $r$  distinct vertices, which remain fixed under various elliptic cycles. Some of them may be disguised if they are fixed by an elliptic element of order 2, because the vertex may have angle  $\pi$ . We still include these vertices. Now consider all vertices that map to a specific vertex under some element of  $\Gamma$ . These each uniquely cover a sort of pie chart  $m$  times where  $m$  is the order of the elliptic elements fixing this vertex. So the sum of the angles of the vertices mapping to a specific vertex is  $2\pi/m$ . If we add all angles at all the elliptic vertices, then we get  $\sum_{i=1}^r \frac{2\pi}{m_i}$ . We may have excluded some (say  $s$ ) cycles of vertices who are not fixed by any elliptic element. So their angles add to  $2\pi$ . The sum of all angles is therefore  $2\pi \left[ s + \sum_{i=1}^r \frac{1}{m_i} \right]$ .

Now, we use the Euler formula  $2 - 2g = (r + s) - e + 1$  for a surface of genus  $g$  consisting of  $(r + s)$  vertices,  $e$  edges, and one face. By triangulating a polygon  $F$  with  $n$  edges and angles  $\alpha_i$ , and applying the Gauss-Bonnet theorem, we see that  $\mu(F) = (n - 2)\pi - \sum \alpha_i$ . Making one more observation, we see that for the orbifold  $\Gamma \backslash \mathbb{H}$ ,  $2e = n$ . Therefore,

$$\mu(\Gamma \backslash \mathbb{H}) = (2e - 2)\pi - \sum \alpha_i = 2\pi \left[ (2g - 2) + \sum_{i=1}^r \left( 1 - \frac{1}{m_i} \right) \right] \quad \blacksquare$$

At this point, I will introduce a theorem which I will not prove, and I will use it to show a corollary to the previous theorem.

**Theorem 2.2** *Poincaré's Theorem If  $g \geq 0$ ,  $r \geq 0$ ,  $m_i$  are integers, and  $2g - 2 + \sum_{i=1}^r \left( 1 - \frac{1}{m_i} \right) > 0$ , then there exists a Fuchsian group with signature  $(g; m_1, \dots, m_r)$ .*

**Corollary 2.3** *For any Fuchsian group  $\Gamma$ ,  $\mu(\Gamma \backslash \mathbb{H}) \geq \pi/21$ .*

**Proof** To be continued...  $\blacksquare$