Final of *Numbers, equations, and proofs*
Theorems and problem sets.

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Hi! This is a 6-hour exam. This means as soon as you open this file your exam starts, and it finishes after six hours. You are not allowed to look at your notes, any book, internet, etc. You are not allowed to talk about the exam to anyone, till you hand in your exam. You will hand in your exam by Friday, Jan 16, 7:00 pm. I will be in my office most of Friday afternoon. Good luck!

1-a) (10 points) State and prove Wolstenholm's theorem.

b) (10 points) Let p be a prime number larger than 3 and m a positive integer number; then $\binom{mp-1}{p-1} \equiv 1 \pmod{p^3}$.

2- (10 points) Let p be an odd prime number, a a positive integer number co-prime to p; then $x^n \equiv a \pmod{p}$ has a solution if and only if $a^{\frac{p-1}{(p-1,n)}} \equiv 1 \pmod{p}$.

3- (13 points) State quadratic reciprocity and Gauss' lemma. Prove Gauss' lemma.

4- (12 points) Let $\xi = \langle a_0, a_1, a_2, \cdots \rangle$. Define two sequences of integers $\{h_n\}$ and $\{k_n\}$ inductively as follows:

$$h_{-2} = 0,\ h_{-1} = 1,\ h_i = a_i h_{i-1} + h_{i-2},\ \text{for } i \geq 0,$$

$$k_{-2} = 1,\ k_{-1} = 0,\ k_i = a_i k_{i-1} + k_{i-2},\ \text{for } i \geq 0.$$

If a and b are positive integer numbers such that

$$|\xi - \frac{a}{b}| < |\xi - \frac{h_n}{k_n}|,$$

for some n, then $b > k_n$. (*Remark: You are allowed to use other theorems and properties of h_n and k_n. However you have to state them properly.*)

5- (10 points) Find all primes p such that $\left(\frac{10}{p}\right) = 1$, where $\left(\frac{\bullet}{\bullet}\right)$ is the Legendre symbol.

6-a) (10 points) If p divides $x^2 + 2$ for some integer x, then $p \equiv 1$ or $3 \pmod 8$.

b) (5 points) Show that there are infinitely many primes of the form $8k + 3$.

7- (10 points) Find the continued fraction of $\sqrt{5}$.

8- (10 points) Let p be a prime of the form $3k + 1$; show that there are integer numbers x and y such that $p = x^2 + xy + y^2$.