MIDTERM I, MATH 103B, WINTER 2012.

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This exam will be considered out of 50 points. (An extra 20 (bonus) points is given.) Good luck!

1. (5 points each) Give either an example or a proof to support your claim.

   (1) There is an integral domain of characteristic 4.
   (2) There is an ideal I of $\mathrm{M}_2(\mathbb{Z})$ such that $\mathrm{M}_2(\mathbb{Z})/I$ is of order 125.
   (3) Let R be a unital ring and assume $1_R + 1_R + 1_R \in U(R)$. Then there is no ideal I such that $R/I \simeq \mathbb{Z}/\mathbb{Z}_3$.
   (4) There is a unital ring R and zero-divisors a and b such that $a + b = 1$.

2. Let $R = \left\{ \begin{bmatrix} a & b \\ 3b & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$.

   (1) (10 points) Prove that R is a commutative subring of $\mathrm{M}_2(\mathbb{Z})$.
   (2) (5 points) Let n be a positive integer and $I_n = \left\{ \begin{bmatrix} a & b \\ 3b & a \end{bmatrix} \mid a, b \in n\mathbb{Z} \right\}$. Prove that I_n is a principal ideal of R.
   (3) (5 points) Find the characteristic of R/I_n.
   (4) (10 points) Prove that R/I_n is isomorphic to $\left\{ \begin{bmatrix} a & b \\ 3b & a \end{bmatrix} \mid a, b \in \mathbb{Z}/n\mathbb{Z} \right\}$. In particular, $R/I_3 \simeq \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}/3\mathbb{Z} \right\}$.
   (5) (5 points) Find the necessary and the sufficient condition for n such that R/I_n is a field.
   (6) (15 points) Let $3 \neq p$ be a prime such that R/I_p is not a field. Prove that $R/I_p \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

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