MIDTERM: MATH 109, FALL 2011.

TIME: 3 HOURS.

Please write and sign that you have honored the UCSD honor code for this exam.

(1) Write the negation of the following propositions.

  (a) (5 points) There is a unique $a \in A, P(a)$.

  (b) (5 points) $(\forall n, P(n) \Rightarrow Q(n)) \vee (\forall n, P(n) \Rightarrow R(n))$.

  (Just saying e.g. $\neg((\forall n, P(n) \Rightarrow Q(n)) \vee (\forall n, P(n) \Rightarrow R(n)))$ is not good enough!)

(2) (7 points) Find a propositional form (a valid combination of $(,), P, Q, R, \wedge, \vee$ and $\neg$) whose truth table is the following

P	Q	R	$*$
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

(3) (a) (5 points) Write in mathematical language what it means to say $\lim_{x\to a} f(x)$ does not exist.

  (b) (8 points) Prove that $\lim_{x\to\sqrt{2}} x^2 = 2$.

(4) (25 points) State the division algorithm and prove it.

Date: 10/28/2011.

(5) Let $f_0 = 0$, $f_1 = 1$ and, for integer n, $f_{n+1} = f_n + f_{n-1}$. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) (5 points) Prove $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$.

(b) (5 points) Prove for any positive integers n, m we have $f_{n+m} = f_{m+1}f_n + f_m f_{n-1}$. (Hint: use $A^{m+n} = A^m A^n$.)

(c) (10 points) Prove for any positive integers n, k we have $f_n | f_{nk}$.

(6) (15 points) Let $a_0 = 2$ and, for any positive integer n, $a_{n+1} = \sqrt{4a_n - 1}$. Prove that a_n is increasing, i.e. for any positive integer n, $a_n \leq a_{n+1}$. (Hint: first prove by induction on n that for any positive integer n we have $2 \leq a_n \leq 2 + \sqrt{3}$.)

(7) For any two subsets A, B of X, let $A \triangle B = (A \cup B) \setminus (A \cap B)$. Let $\chi_A : X \to \{0, 1\}$ be the characteristic function of A, i.e.

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

(a) (5 points) Define recursively what $A_1 \triangle A_2 \triangle \ldots \triangle A_n$ means, where for any $1 \leq i \leq n$, $A_i \subseteq X$.

(b) (10 points) Prove for any $x \in X$ we have $\chi_{A_1 \triangle \ldots \triangle A_n}(x) - \sum_{i=1}^{n} \chi_{A_i}(x)$ is even.

(8) (Bonus problem) Let $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$ and $f : \mathbb{R} \setminus \{-c/d\} \to \mathbb{R}$,

$$f(x) = \frac{ax + b}{cx + d}.$$

(a) (5 points) Is f injective?

(b) (5 points) What is the image of f?

(c) (10 points) Is there an invertible function $\widetilde{f} : \mathbb{R} \cup \{☺\} \to \mathbb{R} \cup \{☺\}$ such that the restriction of $\widetilde{f}$ to $\mathbb{R} \setminus \{-c/d\}$ is equal to f? Prove your answer.