1 (a) Let d be an integer which is at least 2. Prove that for any integer n, either n is not a multiple of d or $n+1$ is not a multiple of d.

(b) Let n_1 and n_2 be two integers such that

$$r_1 n_1 + r_2 n_2 = 1$$

for some integers r_1 and r_2. Prove that for any integer $d > 1$, either $d \nmid n_1$ (d does not divide n_1) or $d \nmid n_2$ (d does not divide n_2).

[**Remark:** (i) Notice that (a) is special case of (b): $(n+1) - n = 1$.

(ii) We did the special case of (a) in class. We proved for any n, either n is odd or $n+1$ is odd.

(iii) Part (b) can be rephrased: n_1 and n_2 have no common divisor larger than 1.

Its "converse" is also true, namely

If n_1 and n_2 have no common divisor larger than 1, then $n_1 r_1 + n_2 r_2 = 1$ for some integers r_1 and r_2.]

2. Let a, b, c and d be real numbers. Prove that

$$(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2).$$

[**Hint**: Construct a proof backwards. Using this method in class, we proved $2xy \leq x^2+y^2$ for real numbers x and y. You are allowed to use this inequality, if needed, without proof.]

3. Prove that the following are valid arguments

(a) $(\neg P) \Rightarrow 0 \vdash P$.

(b) $(P \wedge (\neg Q)) \Rightarrow 0 \vdash P \Rightarrow Q$

[If you want, you can use $P \Rightarrow Q \equiv (\neg P) \vee Q$.]

(c) $P \Rightarrow R\ ,\ Q \Rightarrow R \vdash (P \vee Q) \Rightarrow R$

[**Remark**: As we saw during the class, (a) and (b) are the bases of "proofs by contradicts". For instance (a) says in order to prove P it is enough to show that $(\neg P)$ implies a contradiction. (b) says in order to prove "P implies Q." is true it is enough to show that it is not possible to have P and not Q, i.e. P and $\neg Q$

implies a contradiction.

The case-by-case proofs are based on the argument (c). (c) says if we show each case separately implies the desired conclusion, then the conclusion holds if either of the cases happens.]

4. Prove that, if $a \mid b_1 - b_2$ and $a \mid c_1 - c_2$, then

(a) $a \mid (b_1 + c_1) - (b_2 + c_2)$.

(b) $a \mid b_1 c_1 - b_2 c_2$.

[**Hint**: For (b), this might be useful:

$$b_1 c_1 - b_2 c_2 = b_1 c_1 - b_2 c_1 + b_2 c_1 - b_2 c_2$$
$$= (b_1 - b_2) c_1 + b_2 (c_1 - c_2).$$]

5. Assume for any real number x there is an integer $\lfloor x \rfloor$ such that $\lfloor x \rfloor = \max\{n : \text{integer} \mid n \le x\}$.

$\lfloor x \rfloor$ is called **the integer part of x**. For example

$\lfloor 1.5 \rfloor = 1$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -0.5 \rfloor = -1$, $\lfloor -1.5 \rfloor = -2$.

Prove that

(a) For any real number x,

$$\lfloor x \rfloor = n \iff n \leq x < n+1.$$

(b) If x is a real number which is not an integer, then

$$\lfloor -x \rfloor + \lfloor x \rfloor = -1.$$

(c) For any real number x,

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \tfrac{1}{2} \rfloor.$$

[**Hint**: For part (a), use the fact that there is no integer m such that $n < m < n+1$ if n is an integer.

For part (b), use part (a).

For part (c), let $y = x - \lfloor x \rfloor$. Then by part (a) we have $0 \leq y < 1$. Do a case-by-case proof considering $0 \leq y < 1/2$ or $\frac{1}{2} \leq y < 1$.]

6. Let a, b and c be real numbers. Prove that

$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$$

[**Hint**: Construct a proof backwards. You are allowed to use the following inequality that we proved in class

$a^2+b^2+c^2 \geq ab+ac+bc$.]