1. Prove that, for any positive integer n,

(a) $7 \mid 8^n + 6$,

(b) $7 \mid 6^n + 6 \iff n$ is even.

2. Let $\{a_n\}$ be the following sequence of real numbers

$$a_1 = 1, \qquad a_{k+1} = \frac{6a_k + 5}{a_k + 2} \quad \text{for } k \text{ a positive integer.}$$

Prove that for any positive integer n

(a) $a_{n+1} > a_n$.

(b) $a_n < 5$.

3. Prove that $\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$ for integers $n \geq 2$.

4. Prove that any integer larger than 1 can be written as product of primes.

5. Let X be a set. For any two subsets $A, B \subseteq X$, their symmetric difference is defined by

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

ⓐ Prove that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

ⓑ Prove that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$ for any three subsets $A, B, C \subseteq X$.

ⓒ Prove that $A \triangle A = \emptyset$ for any subset $A \subseteq X$.

ⓓ Prove that $A \triangle \emptyset = A$ for any subset $A \subseteq X$.

6. Let X be a set. Prove that for any two subsets $A, B \subseteq X$ there is a unique subset $C \subseteq X$ such that $A \triangle C = B$.

Hints 1. ⓐ and ⓑ. You can use the previous week's homework assignment which asserts that

$$\left.\begin{array}{l} a \mid b_1 - c_1 \\ a \mid b_2 - c_2 \end{array}\right\} \Rightarrow a \mid b_1 b_2 - c_1 c_2 .$$

Using this result, by induction on n, prove that

$$a \mid b - c \Rightarrow a \mid b^n - c^n .$$

2. Let $f(x) = \dfrac{6x+5}{x+2} = 6 - \dfrac{7}{x+2}$. Prove that

$f(x)$ is increasing and notice that $f(5)=5$.

3. $\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\cdots\left(1-\frac{1}{n^2}\right)$.

Use induction on n.

4. As we did in the class, use strong induction.

5. (a) Use the truth table. [There are other ways to do it.]

(b) Use the truth table. [There are other ways to do it.]

6. Use (5):

$$A \triangle C = B \Rightarrow A \triangle (A \triangle C) = A \triangle B$$

$$\cdots$$

$$A \triangle (A \triangle B) = \cdots .$$