1. Determine if the following functions are 1-1 or onto.

(a) $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $f((a,b)) = 3a - 2b$.

(b) $f: P(X) \to P(X)$, $f(B) = A \,\Delta\, B$ where

X is a non-empty set, $P(X)$ is its power set

and $A \subseteq X$ is a fixed subset of X.

(Hint: Problem 5, HW Due on May 3 and consider $f \circ f$.)

(c) $f: P(X) \to P(A)$, $f(B) = A \cap B$ where

X is a non-empty set, $P(X)$ is its power set

and $\varnothing \neq A \subsetneqq X$ is a fixed proper, non-empty subset of X. (Hint: ① if $A' \subseteq A$, then $A' \cap A = A'$.

② if $x \in X \setminus A$, then $(A \cup \{x\}) \cap A = A$.)

2. Functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are defined as follows.

$$f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ -x & \text{if } -1 \leq x \leq 1 \\ x-2 & \text{if } x > 1 \end{cases}$$

$$g(x) = \begin{cases} x-2 & \text{if} \quad x<-1 \\ -x & \text{if} \quad -1 \leq x \leq 1 \\ x+2 & \text{if} \quad x>1 \end{cases}$$

Find the functions $f \circ g$ and $g \circ f$. Is g the inverse of f?
Is f injective or surjective? How about g?
Sketch and compare the graphs of these functions.

3. Let $f: X \to Y$ and $g: Y \to X$ be two functions. Suppose $f \circ g \circ f = f$ and $g \circ f \circ g = g$. Prove that f is 1-1 $\iff$ g is onto.

(Hint: Use the following result that we proved in the class:
$f: X \to Y$, $g: Y \to X$, $f \circ g \circ f = f$ and $g \circ f \circ g = f$
$f_1: Im(g) \to Im(f)$ and $g_1: Im(f) \to Im(g)$ s.t.
$f_1(x) = f(x)$ and $g_1(y) = g(y)$.
Then $f_1 \circ g_1 = I_{Im(f)}$ and $g_1 \circ f_1 = I_{Im(g)}$. In particular f_1 and g_1 are bijections.)

In the rest of problems, properties of $\vec{f}$ and $\overleftarrow{f}$ are investigated. Let X and Y be non-empty sets and $f: X \to Y$ be a function.

4. (a) Prove or disprove: $\forall\, B_1, B_2 \subseteq Y,\ B_1 \subseteq B_2 \Rightarrow \overleftarrow{f}(B_1) \subseteq \overleftarrow{f}(B_2)$.

(b) Prove or disprove: $\forall\, B_1, B_2 \subseteq Y,\ \overleftarrow{f}(B_1 \cap B_2) = \overleftarrow{f}(B_1) \cap \overleftarrow{f}(B_2)$.

(c) Prove or disprove: $\forall\, A_1, A_2 \subseteq X,\ \overrightarrow{f}(A_1 \cap A_2) = \overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2)$

5. (a) $\forall A \subseteq X,\ A \subseteq \overleftarrow{f}(\overrightarrow{f}(A))$.

(b) $\forall B \subseteq Y,\ \overrightarrow{f}(\overleftarrow{f}(B)) \subseteq B$.

6. $\overrightarrow{f} \circ \overleftarrow{f} \circ \overrightarrow{f} = \overrightarrow{f}$ and $\overleftarrow{f} \circ \overrightarrow{f} \circ \overleftarrow{f} = \overleftarrow{f}$

(Hint: Use Problem 5.)

7. f is injective $\Longleftrightarrow$ $\overrightarrow{f}$ is injective $\Longleftrightarrow$ $\overleftarrow{f}$ is surjective.

(Hint: Use problems 3 and 6.)

8. Let Z be a non-empty set and $g: Y \to Z$ be a function. Prove that

(a) $\overrightarrow{g \circ f} = \overrightarrow{g} \circ \overrightarrow{f}$.

(b) $\overleftarrow{g \circ f} = \overleftarrow{f} \circ \overleftarrow{g}$. (Hint. $x \in \overleftarrow{f}(B) \Longleftrightarrow f(x) \in B$.)