MATH 109: THE SECOND EXAM.
INSTRUCTOR: A. SALEHI GOLSEFIDY

NAME: . . . .

PID: . . . .

(1) Write your Name and PID on the front of your exam sheet.
(2) No calculators or other electronic devices are allowed during this exam.
(3) Show all of your work; no credit will be given for unsupported answers.
(4) Read each question carefully to avoid spending your time on something that you are not supposed to (re)prove.
(5) Ask me when you are unsure if you are allowed to use certain fact or not.
(6) You can choose between problem 3 and problem 4.
(7) There is a bonus problem which is related to problem 4. You do not have to do problem 4 in order to work on the bonus problem.
(8) The bonus problem has 5 extra points.

Problem	Score out of 10
1	
2	
3	
4	
Bonus	

Total=S1+S2+max(S3, S4)+(0.5)×Bonus	....

Date: 05/20/2013.

(1) Let A, B and C be three sets. Prove that
   (a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
   (b) $A \setminus B = \varnothing \Rightarrow A \subseteq B$.

(2) (a) Prove by contradiction that $\forall a, b \in \mathbb{R}, ((\forall \varepsilon \in \mathbb{R}^+, a < b + \varepsilon) \Rightarrow a \leq b)$.
(b) Prove or disprove that $\exists a \in \mathbb{R}, \forall b \in \mathbb{R}, 2 - b^2 \leq a$.

(3) Let $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 5a_n - 6a_{n-1}$ for any positive integer n. Prove that $a_n = 3^n - 2^n$ for any non-negative integer n.

(4) Let n be a non-negative integer. In the game G_n, there is a heap of n stones and each player at her turn removes either *one* or *two* stones. A player wins if she removes the last stone. Assume that both of the players make the best possible moves. Prove that the first player wins if and only if $3 \nmid n$, i.e. n is not a multiple of 3. (Hint: (1) use strong induction on n. (2) you can use the fact that, if $3 \nmid n$, then either $3|n-1$ or $3|n-2$.)

(continue)

(5) (Bonus) Let G_n be the game introduced in Problem 4. Find a necessary and sufficient condition for $(n,m) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ such that the first player wins $G_n \oplus G_m$. (Assume that both of the players make the best possible moves.) (*Try to convey the key idea if you do not have time to present a rigorous proof.*)