MATH 109: THE SECOND EXAM.
INSTRUCTOR: A. SALEHI GOLSEFIDY

NAME: ...Solutions...

PID: ....

1. Write your Name and PID on the front of your exam sheet.
2. No calculators or other electronic devices are allowed during this exam.
3. Show all of your work; no credit will be given for unsupported answers.
4. Read each question carefully to avoid spending your time on something that you are not supposed to (re)prove.
5. Ask me when you are unsure if you are allowed to use certain fact or not.
6. You can choose between problem 3 and problem 4. [redacted]
7. There is a bonus problem which is related to problem 4. You do not have to do problem 4 in order to work on the bonus problem.
8. The bonus problem has 5 extra points.

Problem	Score out of 10
1	
2	
3	
4	
Bonus	

Total=S1+S2+max(S3, S4)+(0.5)×Bonus	....

Date: 05/20/2013.

(1) Let A, B and C be three sets. Prove that
  (a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
  (b) $A \setminus B = \varnothing \Rightarrow A \subseteq B$.

(a) $(x,y) \in A \times (B \cup C) \iff x \in A \wedge y \in (B \cup C)$

$\iff x \in A \wedge (y \in B \vee y \in C)$

$\iff (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$

$\iff (x,y) \in A \times B \vee (x,y) \in A \times C$

$\iff (x,y) \in (A \times B) \cup (A \times C)$.

(b) Suppose to the contrary $A \not\subseteq B \Rightarrow$

$\neg (x \in A \Rightarrow x \in B) \Rightarrow \exists x,\ x \in A \wedge x \notin B$

$\Rightarrow \exists x,\ x \in A \setminus B$

$\Rightarrow A \setminus B \neq \varnothing$ which is a contradiction.

(2) (a) Prove or disprove that $\forall a, b \in \mathbb{R}, ((\forall \varepsilon \in \mathbb{R}^+, a < b + \varepsilon) \Rightarrow a \leq b)$.
(b) Prove or disprove that $\exists a \in \mathbb{R}, \forall b \in \mathbb{R}, 2 - b^2 \leq a$. (III)

(a) True

Suppose to the contrary:

$$\neg(\forall a, b \in \mathbb{R}, ((\forall \varepsilon \in \mathbb{R}^+, a < b + \varepsilon) \Rightarrow a \leq b)$$

$$\equiv \exists a, b \in \mathbb{R}, (\forall \varepsilon \in \mathbb{R}^+, a < b + \varepsilon) \wedge a > b. \quad \text{(I)}$$

Let $c = a - b$. So by (I) we have

$$(\underbrace{\forall \varepsilon \in \mathbb{R}^+, \quad c < \varepsilon}_{\text{(II)}}) \wedge 0 < c$$

However $0 < c \Rightarrow 0 < \frac{c}{2} < c$; and this means $\varepsilon = c/2$ does not satisfy (II), which is a contradiction.

(b) True. Claim. $a = 2$ satisfies (III), i.e.

$$\forall b \in \mathbb{R}, \quad 2 - b^2 \leq 2.$$

Pf of the claim. $\forall b \in \mathbb{R}$, $b^2 \geq 0 \Rightarrow 0 \geq -b^2 \Rightarrow 2 \geq 2 - b^2$.

(3) Let $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 5a_n - 6a_{n-1}$ for any positive integer n. Prove that $a_n = 3^n - 2^n$ for any non-negative integer n.

Proceed by strong induction.

Base. $a_0 = 0 = 3^0 - 2^0$ ✓.

The strong induction step.

$\forall\, 0 \le k \le n,\ a_k = 3^k - 2^k \overset{?}{\Longrightarrow} a_{n+1} = 3^{n+1} - 2^{n+1}$.

Pf of the strong induction step. For $n=1$, clearly equality holds. If $n>1$,

$$
\begin{aligned}
a_{n+1} &= 5a_n - 6a_{n-1} \\
&= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) \quad \text{(by the strong induction hypothesis)} \\
&= (5 \times 3^n - 6 \times 3^{n-1}) - (5 \times 2^n - 6 \times 2^{n-1}) \\
&= 3^{n-1}(15 - 6) - 2^{n-1}(10 - 6) \\
&= 3^{n-1} \times 9 - 2^{n-1} \times 4 \\
&= 3^{n+1} - 2^{n+1}.
\end{aligned}
$$

(4) Let n be a non-negative integer. In the game G_n, there is a heap of n stones and each player at her turn removes either *one* or *two* stones. A player wins if she removes the last stone. Assume that both of the players make the best possible moves. Prove that the first player wins if and only if $3 \nmid n$, i.e. n is not a multiple of 3. (Hint: use strong induction on n.)

We proceed by strong induction on $\underline{n}$.

Base. $n=0 \Rightarrow$ clearly the first player has no win $\Rightarrow G_0$ is an N-game.

The strong induction step

For any $0 \le k < n$,
$3|k \Rightarrow G_k : N$
and
$3 \nmid k \Rightarrow G_k : P$

Strong induction hypothesis

$\overset{?}{\Longrightarrow}$

$3|n \Rightarrow G_n : N$

$3 \nmid n \Rightarrow G_n : P$

$\boxed{3|n}$

Game : G_n
1st : A
2nd : B

Starting Position

The 1st player has two possible moves $\longrightarrow$

Game : G_{n-1} or G_{n-2}
1st : B
2nd : A

The possible positions after one move.

Since $3|n$, $3 \nmid n-1$ and $3 \nmid n-2$. Hence by strong induction hypothe. G_{n-1} and G_{n-2} are P-games. Thus the first player, who is B, can win. So G_n is an N-game.

$\boxed{3 \nmid n}$ Since $3 \nmid n$, either $3|n-1$ or $3|n-2$.

Game : G_n
1st : A
2nd : B

starting position

The 1st player has a move to get to G_m where $3|m$ & $m<n$. $\longrightarrow$

Game : G_m
1st : B
2nd : A
$3|m$

By the strong induction hypothesis, G_m is an N-game, So the 2nd player, who is A, can win. Hence G_n is a P-game.

(5) (Bonus) Let G_n be the game introduced in Problem 4. Find necessary and sufficient condition for $(n, m) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ such that the first player wins $G_n \oplus G_m$. (Assume that both of the players make the best possible moves.)

$G_n \oplus G_m$ is a P-game $\Longleftrightarrow$ $3 \nmid n-m$.

We prove it by strong induction on $m+n$.

The key idea is similar to Problem 4.

- $G_n \oplus G_m$ changes to $G_{n'} \oplus G_{m'} \in \{G_{n-1} \oplus G_m, G_{n-2} \oplus G_m, G_n \oplus G_{m-1}, G_n \oplus G_{m-2}\}$.

  if $3 \mid n-m$, then $3 \nmid n'-m'$.
- If $3 \nmid n-m$, then there is a move to change the game to $G_{n'} \oplus G_{m'}$ s.t. $3 \mid n'-m'$.

You can complete the proof on your own.