

MATH 109: THE SECOND EXAM.
INSTRUCTOR: A. SALEHI GOLSEFIDY

NAME: ... Solutions

PID:

- (1) Write your Name and PID on the front of your exam sheet.
- (2) No calculators or other electronic devices are allowed during this exam.
- (3) Show all of your work; no credit will be given for unsupported answers.
- (4) Read each question carefully to avoid spending your time on something that you are not supposed to (re)prove.
- (5) Ask me when you are unsure if you are allowed to use certain fact or not.
- (6) You can choose between problem 3 and problem 4. XXXXXXXXXXXXXXXXXXXX
- (7) There is a bonus problem which is related to problem 4. You do not have to do problem 4 in order to work on the bonus problem.
- (8) The bonus problem has 5 extra points.

| Problem | Score out of 10 |
|---------|-----------------|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| Bonus | |

| | |
|---|-------|
| Total= $S_1+S_2+\max(S_3, S_4)+(0.5)\times\text{Bonus}$ | |
|---|-------|

Date: 05/20/2013.

(1) Let A, B and C be three sets. Prove that

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) $A \setminus B = \emptyset \Rightarrow A \subseteq B$.

$$\begin{aligned}
 \text{(a) } (x, y) \in A \times (B \cup C) &\iff x \in A \wedge y \in (B \cup C) \\
 &\iff x \in A \wedge (y \in B \vee y \in C) \\
 &\iff (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\
 &\iff (x, y) \in A \times B \vee (x, y) \in A \times C \\
 &\iff (x, y) \in (A \times B) \cup (A \times C).
 \end{aligned}$$

(b) Suppose to the contrary $A \not\subseteq B \Rightarrow$

$$\neg(x \in A \Rightarrow x \in B) \Rightarrow \exists x, x \in A \wedge x \notin B$$

$$\Rightarrow \exists x, x \in A \setminus B$$

$$\Rightarrow A \setminus B \neq \emptyset \text{ which is a contradiction.}$$

- (2) (a) Prove or disprove that $\forall a, b \in \mathbb{R}, ((\forall \varepsilon \in \mathbb{R}^+, a < b + \varepsilon) \Rightarrow a \leq b)$.
 (b) Prove or disprove that $\exists a \in \mathbb{R}, \forall b \in \mathbb{R}, 2 - b^2 \leq a$. (III)

True
 (a) Suppose to the contrary:

$$\neg (\forall a, b \in \mathbb{R}, ((\forall \varepsilon \in \mathbb{R}^+, a < b + \varepsilon) \Rightarrow a \leq b))$$

$$\equiv \exists a, b \in \mathbb{R}, (\forall \varepsilon \in \mathbb{R}^+, a < b + \varepsilon) \wedge a > b. \quad (\text{I})$$

Let $c = a - b$. So by (I) we have

$$(\underbrace{\forall \varepsilon \in \mathbb{R}^+, c < \varepsilon}_{(\text{II})}) \wedge 0 < c$$

However $0 < c \Rightarrow 0 < \frac{c}{2} < c$; and this means

$\varepsilon = c/2$ does not satisfy (II), which is a contradiction.

(b) True. Claim. $a = 2$ satisfies (III), i.e.

$$\forall b \in \mathbb{R}, 2 - b^2 \leq 2.$$

Pf of the claim. $\forall b \in \mathbb{R}, b^2 \geq 0 \Rightarrow 0 \geq -b^2 \Rightarrow 2 \geq 2 - b^2$.

- (3) Let $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 5a_n - 6a_{n-1}$ for any positive integer n . Prove that $a_n = 3^n - 2^n$ for any non-negative integer n .

Proceed by strong induction.

Base. $a_0 = 0 = 3^0 - 2^0 \checkmark$.

The strong induction step.

$$\forall 0 \leq k \leq n, a_k = 3^k - 2^k \stackrel{?}{\Rightarrow} a_{n+1} = 3^{n+1} - 2^{n+1}.$$

pf of the strong induction step. For $n=1$, clearly equality holds. If $n > 1$,

$$a_{n+1} = 5a_n - 6a_{n-1}$$

$$= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) \quad (\text{by the strong induction hypothesis})$$

$$= (5 \times 3^n - 6 \times 3^{n-1}) - (5 \times 2^n - 6 \times 2^{n-1})$$

$$= 3^{n-1}(15 - 6) - 2^{n-1}(10 - 6)$$

$$= 3^{n-1} \times 9 - 2^{n-1} \times 4$$

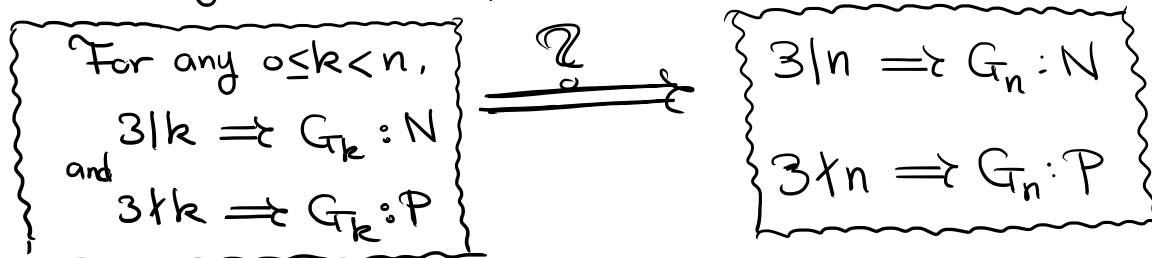
$$= 3^{n+1} - 2^{n+1}.$$

- (4) Let n be a non-negative integer. In the game G_n , there is a heap of n stones and each player at her turn removes either *one* or *two* stones. A player wins if she removes the last stone. Assume that both of the players make the best possible moves. Prove that the first player wins if and only if $3 \nmid n$, i.e. n is not a multiple of 3. (Hint: use strong induction on n .)

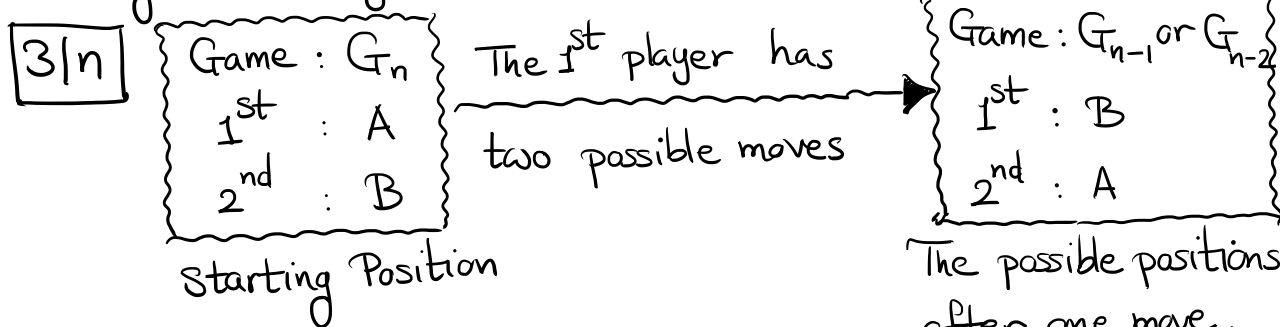
We proceed by strong induction on n .

Base. $n=0 \Rightarrow$ clearly the first player has no win $\Rightarrow G_0$ is an N-game.

The strong induction step

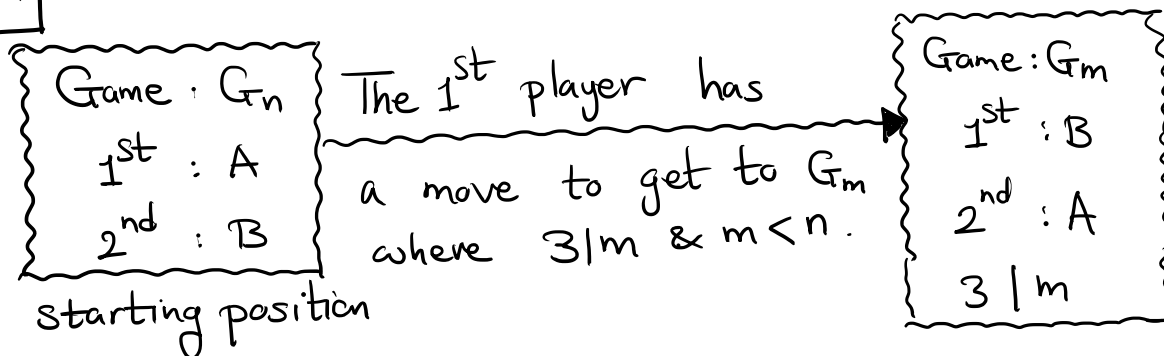


Strong induction hypothesis



Since $3 \mid n$, $3 \nmid n-1$ and $3 \nmid n-2$. Hence by strong induction hypothe. G_{n-1} and G_{n-2} are P-games. Thus the first player, who is B, can win. So G_n is an N-game.

3 <math>\nmid n Since $3 \nmid n$, either $3 \mid n-1$ or $3 \mid n-2$.



By the strong induction hypothesis, G_m is an N-game, so the 2nd player, who is A, can win. Hence G_n is a P-game.

- (5) (Bonus) Let G_n be the game introduced in Problem 4. Find necessary and sufficient condition for $(n, m) \in \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ such that the first player wins $G_n \oplus G_m$. (Assume that both of the players make the best possible moves.)

$$G_n \oplus G_m \text{ is a P-game} \iff 3 \nmid n-m.$$

We prove it by strong induction on $m+n$.

The key idea is similar to Problem 4.

- $G_n \oplus G_m$ changes to $G_{n'} \oplus G_{m'} \in \{G_{n-1} \oplus G_m, G_{n-2} \oplus G_m, G_n \oplus G_{m-1}, G_n \oplus G_{m-2}\}$.
- if $3 \mid n-m$, then $3 \nmid n'-m'$.
- If $3 \nmid n-m$, then there is a move to change the game to $G_{n'} \oplus G_{m'}$ s.t. $3 \mid n'-m'$.

You can complete the proof on your own.