The Modular Group

A **fundamental domain** U for a group of isometries Γ of H^2 is an open, connected subset of H^2 such that

- the intersection of U and any γU is empty, and
- each Γ orbit meets the closure of U, $\overline{U}$.

Fundamental domains can help us better understand the topological spaces that are the fundamental domains' quotient spaces. In Euclidian space, for instance, the interior of the unit square is a fundamental domain, where the isometries are translation by one unit in the two dimensions. "Gluing" the two pairs of opposing edges, we have a quotient space, a torus. (Including a half twist before gluing one of the pairs results in a Klein bottle, another locally Euclidean surface whose fundamental domain is the unit square.) Through their isometries, these fundamental domains tessellate over R^2. The second part of the definition of a fundamental domain is equivalent to the statement that our space X (or in this case, H^2) is equal to

$$\cup\{\gamma\overline{U} : \gamma \in \Gamma\}$$

These fundamental domains are not unique. For instance, creating a small indentation on one side of the square and translating the "missing piece" to the opposite side creates another fundamental domain which behaves exactly the same way.

One example of a fundamental domain on the upper half-plane model is found between the geodesics $|z| = 1$ and $|z| = 2$, where our isometry takes z to 2z. This is easily checked – performing any composition of the isometry other than the identity on any point in the fundamental domain takes it to a point outside the fundamental domain, so we know the first condition holds. Also, we can easily see that the images of the fundamental domain are concentric half-annuli centered at 0, and since these tile H^2 and the boundaries are the images of the boundaries of our fundamental domain, we know that the orbit of any point meets the closure of the fundamental domain.

The classical fundamental domain, Gauss's "modular figure," for the modular group $PSl_2(\mathbb{Z})$ is bounded by the two vertical geodesics at $\frac{1}{2}$ and $-\frac{1}{2}$ and $|z| = 1$. This domain tesselates the entire half-plane with congruent hyperbolic triangles – although the two vertical geodesics do become very close at they approach infinity, they do not form a vertex and are instead a cusp. The quotient space thus has a hole at this almost-vertex and two "corners" where the lower boundary folds over.

Proof

Since the modular group is composed of translations and reflections, we will choose the transformations

$$\sigma(z) = -z^{-1}, \tau(z) = z + 1$$

and τ's inverse, $\omega(z) = z - 1$.

As was previously stated, we can check the second statement of the definition by showing that the triangles tesselate to cover all of the upper half-plane. That is, if a point z has Im(z) > 0, then some composition of isometries will take z to $\overline{M}$.

Lemma: If $Im(z) > 0$, then there are only a finite number of isometries γ for which $Im(\gamma z) \geq Im(z)$. This is because

$$Im(\gamma z) = Im(\frac{az + b}{cz + d}) = Im(\frac{adz + bc\overline{z}}{|cz + d|^2}) = \frac{Im(z)}{|cz + d|^2}$$

So, $|cz+d|$ would have to be less than or equal to 1, which has a finite number of solutions for integers c and d, since cz + d cannot equal 0. Suppose we choose some isometry that maximizes $Im(\gamma z)$. Since translations don't affect the imaginary component at all, we can find such an isometry that γz has $Re[\gamma z] \in [\frac{-1}{2}, \frac{1}{2}]$.

Now we need to show that $|\gamma z| \geq 1$ for it to be in our domain. Let γz be v.

$$Im(v) \geq Im(\frac{-1}{v}) = \frac{Im(v)}{|v|^2}$$

Since Im(v) is positive, we can solve for $|v|$ and find that it is at least 1, and so this point is in the closure of the domain.

Now, to show that the fundamental domain does not intersect with any of its images under Γ, we need to show that for any γz and z in the fundamental domain, γ is the identity. We'll accomplish this by showing that the imaginary component of z has the maximum in z's orbit in the fundamental domain; we will show that if there is more than one member of z's orbit, then their imaginary components are both upper bounds and so are the same.

Recall that

$$Im(\gamma z) = \frac{Im(z)}{|cz+d|^2}$$

Let's consider first the case of c = 0. Then, since the determinant is 1, ad = 1, and so since d has integer values, we have $d^2 = 1$. Since b can be any integer, if b is 0, then we have the identity matrix or negative identity matrix. If b is another integer, the only time we can have more than one member of an orbit in $\overline{M}$ is when they are on the boundaries and not the interior.

Now consider the cases in which c is not 0.

$$\gamma z = \frac{az+b}{cz+d} = \frac{acz+bc}{c(cz+d)}$$

Since ad - bc = 1, we'll substitute bc with ad - 1. So,

$$\gamma z = \frac{acz+ad-1}{c(cz+d)} = \frac{a(cz+d)-1}{c(cz+d)} = \frac{a}{c} + \frac{-1}{c(cz+d)}$$

$$\gamma z - \frac{a}{c} = \frac{-1}{c^2 z + cd} = \frac{-1}{c^2}\frac{1}{z+\frac{d}{c}}$$

$$\left|\gamma z - \frac{a}{c}\right|\left|z+\frac{d}{c}\right| = \frac{1}{c^2}$$

Since a, c, d are integers, and translation doesn't affect the imaginary components, we know that $\left|\gamma z - \frac{a}{c}\right|$ and $\left|z+\frac{d}{c}\right|$ are no smaller than the imaginary components of the vertices of the triangle, which are at $\frac{\sqrt{3}}{2}$. So,

$$\left|\frac{1}{c}\right| \geq \frac{\sqrt{3}}{2}$$

$$|c| \leq \frac{2}{\sqrt{3}}$$

Since c isn't 0, it must be ± 1. But, this means we have

$$|\gamma z \mp a|\,|z \pm d| = 1$$

But, from the conclusion of the first lemma, this does not hold unless $\gamma z = z$.