Midterm MAT 214: New Problems.

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1- (5 points) How many primes, written in base 10, are alternating 1's and 0's beginning and ending with 1?

2- (5 points) Find all positive integers coprime to all terms of $2^n + 3^n + 6^n - 1$. (Hint: $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$!)

3- (5 points) Let $n > 6$ be an integer and $a_1 < a_2 < \cdots < a_{\phi(n)}$ be all the natural numbers less n which are coprime to n. If $a_1, \ldots, a_{\phi(n)}$ is an arithmetic progression, then n is either prime or a power of 2.

4- Let $\{a_n\}_{n=1}^{\infty}$ be the Fibonacci sequence, i.e. $a_0 = 0, a_1 = 1$, and

$$a_{n+1} = a_n + a_{n-1}.$$

a) (5 points) Show that $\text{g.c.d.}(a_{m+1}, a_m) = 1$ for any positive integer m.

b) (10 points) Show that for any positive integer m and non-negative integers q and r, we have

$$a_{mq+r} \equiv a_{m+1}^q a_r \pmod{a_m},$$

and conclude that $\{a_n \pmod{a_m}\}_{n=1}^{\infty}$ is a periodic sequence with shortest period equal to $m \cdot \text{Ord}_{a_m}(a_{m+1})$.

c) (5 points) Show that for any pair of positive integers m and n, we have

$$\text{g.c.d.}(a_m, a_n) = a_d,$$

where $d = \text{g.c.d.}(m, n)$.

5-a) (10 points) Show that $\text{g.c.d.}(b^m - 1, b^n - 1) = b^d - 1$, where $d = \text{g.c.d.}(m, n)$, for any positive integers $b > 1$, m and n.

b) (15 points) Let $b > 1$, $m \neq n$ be positive integers. If $b^n - 1$ and $b^m - 1$ have the same prime divisors, then $b + 1$ is a power of 2.