Midterm MAT 214: Theorems and Problem Sets.

Instructor: Alireza Salehi Golsefidy
8-hour exam

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1-(10 points) Let p be an odd prime and

$$f(x) = \sum_{i=0}^{p-2} (i+1)x^i.$$

Show that if $f(a) \equiv f(b) \pmod{p}$, then $a \equiv b \pmod{p}$.

2-(10 points) Let $a_1 = 1, a_2 = 2, a_3 = 24$, and for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that $n | a_n$ for all positive integers n.

3-(5 points) Let x and y be two positive integers. Show that if $xy | x^2 + y^2 + 1$, then $x^2 + y^2 + 1 = 3xy$.

4- Let $\mathcal{O}_m$ be the ring of integers in $\mathbb{Q}[\sqrt{m}]$.

a) (5 points) If $x, y \in \mathbb{Z}$ and x divides y in $\mathcal{O}_m$, then x divides y in $\mathbb{Z}$.

b) (15 points) Let a and b be two non-zero integers such that $a^2 + 4b \neq 0$ and g.c.d.$(a, b) = 1$. Let $\{x_m\}_{m=0}^{\infty}$ be a sequence of integers as follows:

$$x_0 = 0,\ x_1 = 1, \qquad x_{m+1} = ax_m + bx_{m-1}.$$

Show that g.c.d.$(x_m, x_n) = x_{\text{g.c.d.}(m,n)}$ for all positive integers m and n. (Hint: First show that $x_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}$, where α and β are solutions of $x^2 - ax - b = 0$.)

5-(15 points) Find all positive integers n such that $n = x^2 + xy + y^2$ for some integers x and y. If p is prime and $p = x^2 + xy + y^2$, then at most how many integer solutions does the equation $p = x^2 + xy + y^2$ have? (Hint: use $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$.)