Ricci flow on Wallach flag varieties

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January 19,2013

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- This lecture is an exposition of joint work with Man Wai (Mandy) Cheung.

The curvature

- If $x_1 = x_2$ then the sectional curvature is is strictly positive if $0 < \frac{x_3}{x_1} < 1$ or $1 < \frac{x_3}{x_1} < \frac{4}{3}$ and there is some strictly negative curvature if $\frac{x_3}{x_1} > \frac{4}{3}$

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- The symmetric group acting by permuting facters preserves positive curvature. We consider the case when $x_3 < x_1 < x_2$. Since scaling by a constant preserves the sign of curvature we consider $x_1 = 1, x_2 = 1 + r$ and $x_3 = s$ with $r > 0$ and $0 < s < 1$.

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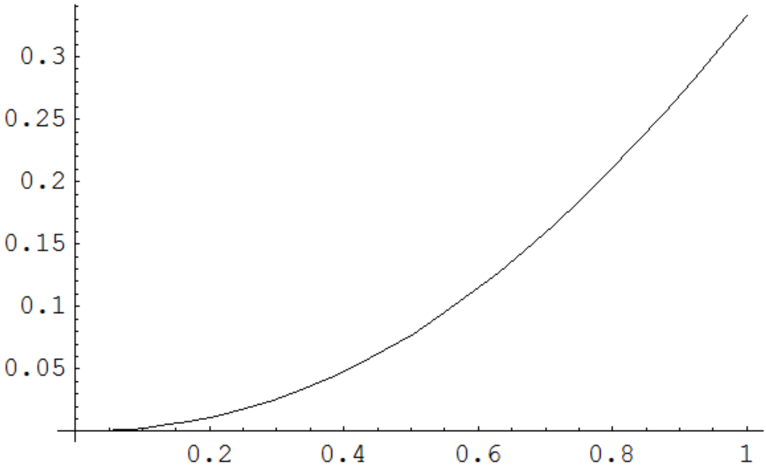
- We note that if $0 < s < 1$ then

$$\frac{s^2}{4} < \frac{s-2+2\sqrt{1-s+s^2}}{3} < \frac{s^2}{3}.$$



Fundamental domain for S_3 acting on the homogeneous metrics of positive curvature consists of the points in the first quadrant below the graph the sets $\{(s,0)|\ 0 < s < 1\}$ and $\{(1,r)|0 < r < \frac{1}{3}\}$.

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- Hamilton's Ricci flow is given in these parameters as
  $$\frac{dx_i}{dt} = -2r_i x_i.$$
- The goal is to say what happens to positive sectional curvature or Ricci curvature under the above non-linear ODE.

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- A direct calculation shows that for $d = 2, 4, 8$

$$\frac{d}{dt}\frac{x_3(t)}{x_1(t)} = -2\frac{x_3(t)}{x_1(t)}(r_3 - r_1) = \frac{-2d(1-\frac{x_3}{x_1})(4\frac{(d-1)}{d} - \frac{x_3}{x_1})}{x_1^2}.$$

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- Hence if $0 < \frac{x_3(t)}{x_1(t)} < 1$ then $\frac{d}{dt}\frac{x_3(t)}{x_1(t)} < 0$, if $1 < \frac{x_3(t)}{x_1(t)} < 4\frac{d-1}{d}$ then $\frac{d}{dt}\frac{x_3(t)}{x_1(t)} > 0$ and if $\frac{x_3(t)}{x_1(t)} > 4\frac{d-1}{d}$ then $\frac{d}{dt}\frac{x_3(t)}{x_1(t)} < 0$. That is the line through $1, 1, 1$ is repelling fixed point and that through $1, 1, 4\frac{d-1}{d}$ is an attractor.

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- The lines through $1, 1, 1$ and $1, 1, 4\frac{d-1}{d}$ give the full set of Einstein metrics among the metrics with $x_1 = x_2$.

- This implies that if $1 < \frac{x_1(0)}{x_3(0)} < 4\frac{d-1}{d}$ then we have $\lim_{t\to+\infty} \frac{x_1(t)}{x_3(t)} = 4\frac{d-1}{d}$ under the Ricci flow.

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- We also note that since the Ricci tensor is positive definite for $1, 1, s$ and $0 < s \leq 4\frac{d-1}{d}$ this implies that the flow cannot change the signature of the Ricci tensor if it starts with strictly positive curvature and $x_1 = x_2$.

Ricci curvature

- We assume $x_2 > x_1 > x_3 > 0$ and scale to $x_1 = 1, x_2 = 1 + r, x_3 = s$ with $r > 0$ and $0 < s < 1$.

$$r_1 x_1 = \frac{-2rd - dr^2 + (10d-8)s + (10d-8)rs - ds^2}{2(1+r)s},$$

$$r_2 x_2 = \frac{dr + dr^2 + (10d-8)s - ds^2}{2s}$$

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- If $0 < r < 8$ only the first can change sign: positive definite Ricci curvature if and only if

$$r < \sqrt{1 + 8s^2} - (1 - 3s), d = 2$$

$$r < \sqrt{1 + 15s^2} - (1 - 4s), d = 4$$

$$r < \sqrt{1 + \frac{77}{4}s^2} - (1 - \frac{9}{2}s), d = 8.$$

since all of these expressions are < 8 if $0 < s < 1$.

- To change the signature we start with a point with $r_1 = 0$ and hope that $\frac{dr_1}{dt} = -2\sum r_i x_i \frac{\partial r_1}{\partial x_i} < 0$. This works for $d = 2, 4, 8$ respectively if

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- $$0 < s < 1 - \sqrt{\frac{5}{8}} = 0.20943058...$$
  $$0 < s < \frac{30 + 5\sqrt{21} - 3\sqrt{5(21 + 4\sqrt{21})}}{30} = 0.361437...$$
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For all the examples the Ricci flow of a metric with positive definite Ricci tensor can flow to one with signature $(d, 2d)$.

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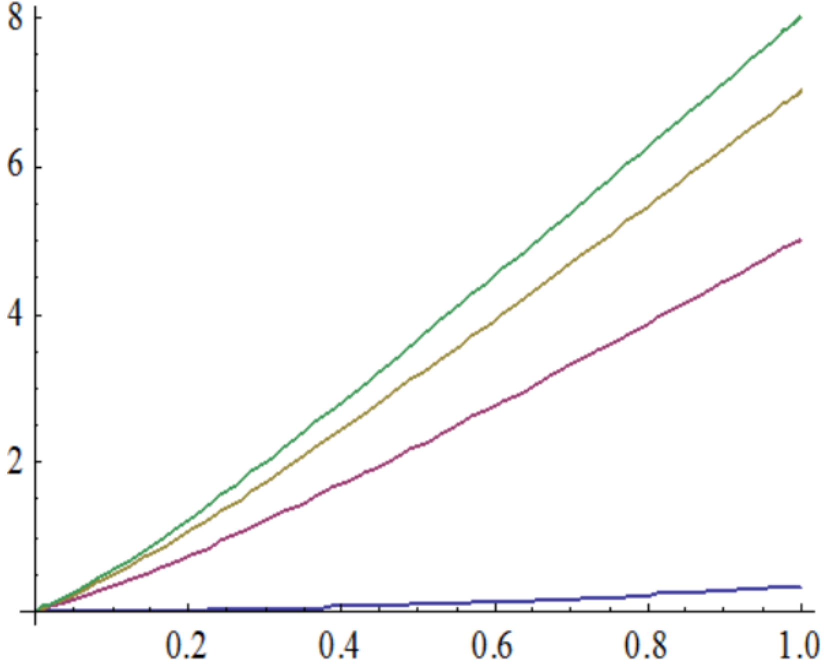
Theorem

If g_o is a homogeneous Riemannian structure on the 6 dimensional example with strictly positive sectional curvature then under the Ricci flow it retains strictly positive Ricci curvature.


8
6
4
2
0.2
0.4
0.6
0.8
1.0

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- $$s' = \frac{-2(x_3' x_1 - x_1' x_3)}{x_1^2} = h(d, r, s).$$
- $$g(d, r, s) = \begin{cases} -4\frac{r}{s}(2 + r - 3s), d = 2 \\ -8\frac{r}{s}(2 + r - 4s), d = 4 \\ -8\frac{r}{s}(4 + 2r - 9s), d = 8 \end{cases} .$$

- $$h(d,r,s)=\begin{cases} 4\frac{1-s}{1+r}(-2-3r+s), d=2 \\ 8\frac{1-s}{1+r}(-3-4r+s), d=4 \\ 8\frac{1-s}{1+r}(-7-9r+2s), d=8 \end{cases}.$$

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- If $0 < s < 1$ and $r > 0$ then $h(d,r,s) < 0$. We can thus think of r as a function of s in this range and have

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$$r'(s) = \frac{r'(t)}{s'(t)} = \frac{r}{s} f(d,r,s)$$
- $$f(d,r,s) = \frac{g(d,r,s)}{h(d,r,s)} = \frac{1+r}{1-s} \begin{cases} \frac{2+r-3s}{2+3r-s}, d=2 \\ \frac{2+r-4s}{3+4r-s}, d=4 \\ \frac{4+2r-9s}{7+9r-2s}, d=8 \end{cases}$$

Lemma

Suppose that we have a solution to the Ricci flow with initial condition $s_o > 0, r(s_o) > 0$ *and* $r(s)$ *is defined for* $0 < s_1 \leq s \leq s_o$.

1. If $f(d, r(s), s) \geq C > 0$ *in this range then we have*

$$r(s) \leq s^C \frac{r(s_o)}{s_o^C}, s_1 \leq s \leq s_o.$$

2. If $0 < f(d, r(s), s) \leq C$ *in this range we have*

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Lemma

If $d = 2$ *then* $r_2, r_3 > 0$ *if* $0 < s < 1$ *and* $0 < r < 2(1 + \sqrt{2})$.

Lemma

If $d = 2, 0 < s < 1$ *and* $r(s) > s$ *then* $r'(s) > 0$. *Suppose that* $0 < s_o < 1$, $s_o < r(s_o) \leq 2s_o$ *and* $0 < s_1 < s_o$ *is such that* $r(s)$ *is defined and* $r(s) > s$ *for* $s_1 \leq s \leq s_o$. *Then* $r(s) < 2s$.

The point here is that the smallest value of C in the calculus lemma is 1.

- We have seen that the condition for some negative Ricci curvature is

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- The Lemma above implies that under this condition $r(s)$ can never pass $2s$.
- This completes the argument for the case $d = 2$.

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- Our argument (as in the case of Bohm-Wilking) works only when s is very small.
- One finds that in these cases one can take $C = \frac{5}{6}$ so

$$r(s) \geq s^{\frac{5}{6}} Const.$$

for s sufficientlly small and since $s \longrightarrow 0$ along the Ricci flow hence along the flow $\frac{r}{s}$ becomes arbitrarily large.