Noncoherence of lattices

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Definitions and examples

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- Free-by-cyclic groups (Feighn and Handel).
- Certain classes of small cancellation groups (McCammond and Wise). For instance, $G = \langle x_1, ..., x_n | W^m \rangle$, where $m > |W|$.

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Problem

(Serre, 1977) Are $SL(3, \mathbf{Z})$ and $SL(2, \mathbf{Z}(\frac{1}{p}))$ noncoherent?

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Let G be a connected semisimple Lie group without compact factors and G not locally isomorphic to $SL(2, \mathbf{R})$ and $SL(2, \mathbf{C})$. Then every lattice Γ in G is non-coherent.

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Note that lattices in $SL(2, \mathbf{R})$ and $SL(2, \mathbf{C})$ are (virtually) free, surface and 3-manifold groups, so they are coherent.

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Theorem

Let $\Gamma < SO(n, 1)$ be an arithmetic lattice of the simplest type (associated with a quadratic form over a number field). Then Γ is noncoherent provided that $n \geq 4$. (Kapovich, Potyagailo, Vinberg; Agol)

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Theorem

Let $\Gamma < SO(n,1)$ be an arithmetic lattice of quaternionic type (associated with a hermitian quadratic form over a central 4-dimensional division ring). Then Γ is non-coherent provided that $n \geq 4$.

Corollary

Every arithmetic lattice $\Gamma < SO(n,1)$ *is noncoherent provided that* $n \geq 4$, $n \neq 7$.

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Observation

All known constructions of non-arithmetic lattices in $SO(n,1)$, $n \geq 4$ (Makarov; Gromov-Piatetsky-Shapiro; Agol) lead to noncoherent groups. (Kapovich, Potyagailo, Vinberg)

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Observation

All known examples of non-arithmetic lattices in $SU(n,1), n = 2,3$ are noncoherent.

Theorem

Every lattice in Isom$(\mathbf{HH}^n)$ *and Isom*$(\mathbf{OH}^2)$ *is noncoherent.*

Proof: Reduction to the $SO(4,1), SO(8,1)$ cases.

Proof in the complex-hyperbolic case

Assume Γ is torsion-free. Since $H^1(\Gamma) \neq 0$ and Γ is a Kähler group,

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Γ contains $\mathbf{Z}^2$.

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Γ contains $\mathbf{Z}^2$. (Essentially due to Birman–Lubotzky–McCarthy) Describe holomorphic fibration in the blown up "complete quadrangle" case if times permits.

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More on this if time permits.

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We do not even know if $\mathbf{Z}^2 \star \mathbf{Z}$ embeds in $SL(3, Z)$!