Parabolic and Borel subgroups

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Def. A variety X is called <u>complete</u> if for any variety Y

$$X \times Y \xrightarrow{Pr} Y \quad \text{is closed.}$$

Basic properties of complete varieties.

Suppose X is complete. Then

① Any closed subvariety of X is complete.

② If Y is complete, then $X \times Y$ is complete.

③ $X \xrightarrow{\phi} Y$ a morphism $\Rightarrow$ $\phi(X)$ is complete, and closed

④ $X \subseteq Y$ subvariety $\Rightarrow$ X is closed.

⑤ X : irreducible $\Rightarrow$ $\mathcal{O}_X(X) = k$.

⑥ X : affine $\Rightarrow$ $|X| < \infty$.

Pf. ① Let $X' \subseteq X$ be a closed subvariety. Then

$$\begin{tikzcd}
X' \times Y \arrow[r] \arrow[d, "\text{closed}"'] \arrow[dr, phantom, "\circlearrowleft"] \arrow[dr, bend right=60, "\text{closed}"'] & Y \arrow[d, "="] \\
X \times Y \arrow[r, "\text{closed}"'] & Y
\end{tikzcd}$$

②
$$\begin{tikzcd}
X \times Y \times Z \arrow[r, "\text{closed}"'] \arrow[rr, bend right, "\text{closed}"'] & Y \times Z \arrow[r, "\text{closed}"'] & Z
\end{tikzcd}$$

③ $$X \times Y \xrightarrow{\text{closed}} Y$$

③

$$\begin{array}{ccc} X \times Y & \xrightarrow{\text{closed}} & Y \\ \text{closed} \ \cup\!| & \circlearrowright & \cup\!| \ \text{closed} \\ \Gamma_\phi := \{(x, \phi(x)) \mid x \in X\} & \longrightarrow & \phi(X) \end{array}$$

$$\left.\begin{array}{ccccc} X & \longrightarrow & \Gamma_\phi & \longrightarrow & X \\ x & \longmapsto & (x, \phi(x)) & \longmapsto & x \end{array}\right\} \Rightarrow \Gamma_\phi \simeq X \text{ is complete.}$$

$$\begin{array}{ccc} \psi^{-1}(C) & \longrightarrow & pr_Z(\psi^{-1}(C)) \quad \text{closed} \\ \subseteq_{\text{closed}} \ X \times Z & \longrightarrow & Z \\ \psi := (\phi, id.) \downarrow & & \downarrow \quad \| \, ? \\ \phi(X) \times Z & \longrightarrow & Z \\ C \subseteq_{\text{closed}} & \longrightarrow & pr_Z(C) \end{array}$$

$pr_Z(C) \ni z \Rightarrow (\phi(x), z) \in C$

$\Rightarrow (x, z) \in \psi^{-1}(C)$

$\Rightarrow z \in pr_Z(\psi^{-1}(C))$.

④ $X \overset{\phi}{\hookrightarrow} Y \Rightarrow \phi(X) = X$ is closed by $\underline{3}$.

⑤ $f \in \mathcal{O}_X(X) \Rightarrow f(X)$ is closed in $\mathbb{A}^1$, and irreducible

$\Rightarrow$ either $f(X) = \mathbb{A}^1$ or $|f(X)| = 1$.

$$\underbrace{\underset{\text{closed in } \mathbb{A}^1 \times \mathbb{A}^1}{\{(x, y) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid xy = 1\}} \xrightarrow[pr]{} \underset{\text{NOT closed in } \mathbb{A}^1}{\{y \in \mathbb{A}^1 \mid y \neq 0\}}}$$

$\Downarrow$

$\mathbb{A}^1$ is NOT complete.

Since $f(X)$ is complete, we get that $f(X) \neq \mathbb{A}^1$.

So f is constant.

⑥ X is affine $\Rightarrow |X| = |\operatorname{Max}(k[X])| = |\operatorname{Max}(\mathcal{O}_X(X))| < \infty$.

because

$\boxed{\underline{\text{Theorem}}.\ \text{A projective variety is complete.}}$

<u>Theorem.</u> C : irreducible smooth $\underbrace{\text{curve}}_{\dim = 1 \text{ variety}}$

$U \subseteq C$ non-empty open set

X : complete ; $\phi : U \longrightarrow X$ a morphism

$\Rightarrow \exists\ \tilde{\phi} : C \longrightarrow X$ s.t. $\tilde{\phi}|_U = \phi$.

<u>Def.</u> A closed subgroup P of an affine algebraic gp is called parabolic if G/P is a complete variety.

<u>Basic properties of parabolic subgps.</u>

① If $P \leq G$ is parabolic, then G/P is projective.

② If $P \leq G$ is parabolic and $P \leq \underbrace{Q}_{\text{closed subgp}} \leq G$, then Q is parabolic.

③ $\left.\begin{array}{l} P \leq G \text{ parabolic} \\ Q \leq P \text{ parabolic} \end{array}\right\} \Longrightarrow Q \leq G$ parabolic.

④ $P \leq G$ parabolic $\Longleftrightarrow P^\circ \leq G^\circ$ parabolic.

<u>Pf.</u> ① G/P is a quasi-projective variety $\Rightarrow G/P \hookrightarrow \mathbb{P}^n$ as a subvariety

G/P is complete

$G/P \Longleftarrow G/P$ is closed

$$G/P \text{ is projective.} \quad \Longleftarrow \quad G/P \text{ is closed in } \mathbb{P}^n \qquad \dashv$$

② $G/P \twoheadrightarrow G/Q$

complete $\quad$ complete

③ $G/Q \times X \xrightarrow{\psi} G/P \times X \longrightarrow X$

$\quad$? closed $\qquad$ closed

For $Z \subseteq G/Q \times X$ closed, let $\tilde{Z} := \{(g,x) \in G \times X \mid (gQ, x) \in Z\}$.

So $\tilde{Z} \subseteq G \times X$ is closed. Since
$$P \times G \times X \longrightarrow G \times X$$
$$(p, g, x) \longmapsto (gp, x)$$
is a morphism, we have

$$\tilde{Z}_P := \{(p,g,x) \in P \times G \times X \mid (gp, x) \in \tilde{Z}\} \text{ is closed.}$$

Since $\quad P \times G \times X \underset{\text{open}}{\longrightarrow} P/Q \times G \times X \underset{\text{closed}}{\longrightarrow} G \times X \underset{\text{open}}{\longrightarrow} G/P \times X$,

we have $\quad \tilde{Z}_P \overset{?}{\longmapsto}$ closed $\longmapsto \{(g,x) \mid \exists p \in P,\ (gp, x) \in \tilde{Z}\}$ closed

$\longmapsto \{(gP, x) \mid \exists p \in P,\ (gpQ, x) \in Z\}$ closed

$= \psi(Z)$ closed.

④ G/G° : finite $\Rightarrow$ $G^\circ \leq G$ parabolic

and $P^\circ \leq P$ parabolic

- If $P \leq G$ parabolic $\Rightarrow$ $P^\circ \leq G$ parabolic

$$\Rightarrow \left.\underset{\text{closed}}{G^\circ/P^\circ} \hookrightarrow \underset{\text{complete}}{G/P^\circ}\right\} \Rightarrow P^\circ \leq G^\circ \text{ parabolic.}$$

- If $P^\circ < G^\circ$ parabolic $\Rightarrow$ $P^\circ \leq G$ parabolic

$$\Rightarrow P \leq G \text{ parabolic.}$$

<u>Lemma</u>. Let $B :=$ upper-triangular inv. matrices. Then B is a parabolic subgp of $GL_n(k)$.

<u>Pf</u>. $P_0 = GL_n(k) \supseteq P_1 := \left\{ \begin{bmatrix} * & * \\ & * \end{bmatrix} \right\} \supseteq P_2 = \left\{ \begin{bmatrix} * & * & * \\ & * & \\ & & * \end{bmatrix} \right\} \supseteq \cdots \supseteq P_n = B.$

Then P_i's are closed subgroups of $GL_n(k)$.

$\forall i$, $P_i(V_j) = V_j$, for $j \leq i$, where $V_i = \langle e_1, \ldots, e_i \rangle$.

$\Rightarrow P_i \underset{\text{transitively}}{\curvearrowright} \mathbb{P}(V_i)$ and $\mathrm{Stab}_{P_i}([e_i]) = P_{i+1}$.

$\Rightarrow P_i/P_{i+1} \xrightarrow[\substack{\text{bijective}\\ \text{morphism}}]{} \mathbb{P}(V_i)$ and $P_i/P_{i+1} \times X \xrightarrow[\substack{\text{open}\\ \text{bijective}\\ \Downarrow \\ \text{closed}}]{} \mathbb{P}(V_i) \times X$

$\Rightarrow P_i/P_{i+1}$ is complete $\Rightarrow P_{i+1} \leq P_i$ parabolic

$\Rightarrow B \leq GL_n(k)$ is parabolic. ∎

$$\boxed{\text{\underline{Theorem}. A connected affine algebraic group } G \text{ contains a proper algebraic group if and only if } G \text{ is NOT solvable.}}$$

<u>Corollary</u> (Borel's fixed point theorem)

G : connected solvable $\Big\} \Rightarrow X^G \neq \varnothing$.

$$\left.\begin{array}{l} G:\text{ connected solvable} \\ X:\text{ complete variety} \\ G \curvearrowright X \end{array}\right\} \Rightarrow X^G \neq \emptyset. \quad (\text{fixed points}).$$

Pf. G has a closed orbit $G\cdot x_0 \Rightarrow G/G_{x_0} \xrightarrow{\text{open}} G\cdot x_0$ bijective morphism and $G\cdot x_0$ is complete.

$$\begin{tikzcd} G/G_{x_0} \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\parallel"] \\ G\cdot x_0 \times Y \arrow[r] & Y \end{tikzcd}$$

$\Rightarrow G/G_{x_0}$ is complete

$\Rightarrow G_{x_0} \leq G$ is parabolic

$\Rightarrow$ by the previous theorem $G_{x_0} = G \Rightarrow x_0 \in X^G$. ∎

Corollary (Lie–Kolchin theorem)

Let $G \subseteq GL_n(k)$ be a connected solvable group. Then $\exists\, x \in GL_n(k)$ s.t. $x^{-1} G x \subseteq \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}$ (upper-triangularized.)

Pf. $G \curvearrowright \underbrace{GL_n(k)/B}_{\text{complete by Lemma}} \Rightarrow$ it has a fixed point xB

$\Rightarrow x^{-1} G x \subseteq B$. ∎

Pf of theorem. ($\Leftarrow$) Suppose $G \subseteq GL(V)$ is NOT solvable, and it does NOT have a proper parabolic.

$G \curvearrowright \mathbb{P}(V)$ has a closed orbit $G\cdot v_1 \Rightarrow$

$$\left.\begin{array}{l} G/G_{v_1} \longrightarrow G\cdot v_1 \text{ bijective } G\text{-equi. morphism} \\ G\cdot v_1 \text{ complete} \end{array}\right\} \Rightarrow G_{v_1} \leq G \text{ parabolic}$$

$$\Rightarrow G = G_{v_1}.$$

$\Rightarrow G \curvearrowright \mathbb{P}(V/\langle v_1\rangle)$. Repeating this argument we get a full flag $\langle v_1\rangle \subsetneq \langle v_1, v_2\rangle \subsetneq \cdots \subsetneq \langle v_1, \ldots, v_n\rangle$ which is G-invariant.

$\Rightarrow \exists\, x \in GL_n(k)$ s.t. $x^{-1}Gx \subseteq B \Rightarrow G$ is solvable, which is a contradiction.

($\Rightarrow$) Suppose G is solvable and it has a proper parabolic subgp P. Suppose P is of maximum possible dimension among proper parabolic subgroups. $P^\circ \leq G$ is parabolic of the same dim. as P. So we can assume P is connected.

<u>Claim</u>. $[G,G] := \langle g_1 g_2 g_1^{-1} g_2^{-1} \mid g_1, g_2 \in G\rangle$ is a closed connected subgp of G (if G is a connected affine algebraic gp.) (prove it later)

So $\dim [G,G] < \dim G$, $[G,G]$ is a connected solvable group.

$\Rightarrow P\,[G,G]$ is a connected closed subgroup of G

$\Rightarrow P\,[G,G]$ is a parabolic subgp of G

So by maximality of dim P, we get that either $[G,G] \subseteq P$ or $G = P[G,G]$.

<u>$[G,G] \subseteq P$ cannot happen.</u>

If $[G,G] \subseteq P$, then $P \triangleleft G \Rightarrow G/P$ is an affine algebraic gp

G/P affine and complete and irreducible $\Rightarrow |G/P| = 1 \Rightarrow$

$P = G$ which is a contradiction.

So <u>$G = P[G,G]$</u>.

Hence $[G,G]/P \cap [G,G] \longrightarrow G/P$

is a bijective $[G,G]$-equiv. map

$\Rightarrow P \cap [G,G]$ is a parabolic in $[G,G]$

By the induction hypothesis, $P \cap [G,G] = [G,G] \Rightarrow$

$[G,G] \subseteq P$ which is a contradiction. ∎

<u>Pf of the above claim.</u>

$\forall g \in G$, let $\phi_g : G \to G$, $\phi_g(h) := g h g^{-1} h^{-1}$.

$\Rightarrow \phi_g(G)$ is an irreducible subset of G, which contains e,

Let $Y_{g_1,\dots,g_n;\varepsilon_1,\dots\varepsilon_n} := \overline{\phi_{g_1}(G)^{\varepsilon_1} \cdot \ldots \cdot \phi_{g_n}(G)^{\varepsilon_n}}$

$\Rightarrow Y_{g_1,\dots,g_n;\varepsilon_1,\dots,\varepsilon_n}$ is closed and irreducible.

Ex. Show that

$$Y_{g_1,\dots,g_n;\varepsilon_1,\dots,\varepsilon_n} \cdot Y_{h_1,\dots,h_m;\delta_1,\dots,\delta_m} \subseteq Y_{g_1,\dots,g_n,h_1,\dots,h_m;\varepsilon_1,\dots,\varepsilon_n,\delta_1,\dots,\delta_m} \quad (*)$$

Suppose $Y_{g_1,\dots,g_n;\varepsilon_1,\dots,\varepsilon_n}$ has the maximum dim. among these sets.

$\Rightarrow$ by $(*)$ we get that

① $H := Y_{g_1,\dots,g_n;\varepsilon_1,\dots,\varepsilon_n}$ is a closed subgp

② $\forall g \in G$, $\phi_g(G) \subseteq H$.

So
$$\begin{aligned} G\times\dots\times G &\xrightarrow{\psi} H \\ (h_1,\dots,h_n) &\longmapsto \phi_{g_1}(h_1)^{\varepsilon_1}\cdots\phi_{g_n}(h_n)^{\varepsilon_n} \end{aligned}$$
is a dominant morphism.

$\Rightarrow$ $\operatorname{Im}\psi$ contains a dense open subset of H

$\Rightarrow$ $\operatorname{Im}\psi \cdot \operatorname{Im}\psi = H \Rightarrow H = \phi_{g_1}(G)^{\varepsilon_1}\cdots\phi_{g_n}(G)^{\varepsilon_n}\cdot\phi_{g_1}(G)^{\varepsilon_1}\cdots\phi_{g_n}(G)^{\varepsilon_n}$

$\Rightarrow$ $H \leq [G,G] = \langle \phi_g(G) \mid g\in G\rangle \subseteq H$.

$\Rightarrow$ $[G,G]$ is a closed connected subgroup of G.

From abstract group theory we know that it is normal. ∎

Def. A maximal connected solvable subgroup of G is called a Borel subgroup of G.

Theorem. ① A Borel subgroup is parabolic.

② Any parabolic subgroup P of G contains a Borel subgroup of G.

Theorem. ① A Borel subgroup is parabolic.

② Any parabolic subgroup P of G contains a Borel subgroup of G.

③ Any two Borel subgroups of G are conjugate.

Pf. • Suppose G is a closed subgroup of $GL_n(k)$

$\Rightarrow G \curvearrowright \underbrace{GL_n(k)/T_r}_{\text{flag variety}}$ has a closed orbit

$\Rightarrow \exists\, x_0 = gT_r$ s.t. G/G_{x_0} is complete

and $G_{x_0} \subseteq g T_r g^{-1}$

$\Rightarrow G_{x_0}$ is parabolic and solvable

$\Rightarrow G_{x_0}^{\circ}$ is parabolic and solvable

$\Rightarrow \exists$ a Borel subgroup B_0 which is parabolic.

• Let B be a Borel subgroup. $\Rightarrow B \curvearrowright G/B_0$ has a fixed point

$\Rightarrow \exists\, g \in G$ s.t. $B \subseteq g B_0 g^{-1}$

Since B and gB_0g^{-1} are both irreducible and of the same dimension, $B = gB_0g^{-1} \Rightarrow B$ is parabolic.

This proves ① and ③.

• Let P be parabolic in G. Let B_P be a Borel subgroup of P. Then B_P is parabolic in P. So B_P is parabolic in G.

Let B_G be a Borel subgroup of G. Then $B_P \curvearrowright G/B_G$ has a fixed point $\Rightarrow B_P \subseteq g B_G g^{-1}$ for some $g \in G$.

So w.l.o.g. we can assume $B_P \subseteq B_G \Rightarrow$

$B_G/B_P \subseteq G/B_P$ is closed and so it is complete

$\Rightarrow B_P$ is parabolic in $B_G \Rightarrow B_P = B_G$.

$\Rightarrow B_G \subseteq P$. ∎