Commutative algebraic groups

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Theorem. G : commutative affine algebraic group

$$G_s := \{ g \in G \mid g : \text{semisimple} \}$$

$$G_u := \{ g \in G \mid g : \text{unipotent} \}$$

① G_s and G_u are closed subgroups

② $G_s \times G_u \longrightarrow G \quad (s,u) \longmapsto su$ is an isomorphism.

Pf ① Fix an embedding $G \subseteq GL_n(k) \Rightarrow G_s = \{ g \in G \mid \text{diagonalizable in } GL_n(k) \}$

and $G_u = \{ g \in G \mid g \in GL_n(k) \text{ is unipotent.} \}$

$G_s \subseteq GL_n(k)$ consists of pairwise commuting semisimple

elements $\Rightarrow \exists\, x \in GL_n(k)$ s.t. $x G_s x^{-1} \subseteq$ diag. matrices

$$\Rightarrow x \overline{\langle G_s \rangle} x^{-1} \subseteq \text{diag. matrices}$$

$$\Rightarrow \overline{\langle G_s \rangle} \subseteq G_s$$

$\Rightarrow G_s = \overline{\langle G_s \rangle} \Rightarrow G_s$ is a closed subgroup of G.

- G_u consists of commuting matrices $\Rightarrow \exists\, y \in GL_n(k)$ s.t.

$$\left.\begin{array}{l} y G_u y^{-1} \subseteq \text{upper\_triang. matrices} \\ G_u : \text{consists of unipotent elements} \end{array}\right\} \Rightarrow y G_u y^{-1} \subseteq \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix} \right\}$$

$$\Rightarrow y \overline{\langle G_u \rangle} y^{-1} \subseteq \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix} \right\}$$

$\Rightarrow \overline{\langle G_u \rangle} \subseteq G_u \Rightarrow G_u$ is a closed subgroup of G.

② Since G, G_u, and G_s are abelian groups,

$$\phi : G_s \times G_u \longrightarrow G, \quad \phi(s,u) := su$$

is an algebraic group homomorphism.

- $(s,u) \in \ker \phi \Rightarrow s = u^{-1} \Rightarrow s$ is both semisimple and unipotent

$$\Rightarrow s = I \Rightarrow u = I.$$

- Because of Jordan decomposition, ϕ is surjective.

- Since G is abelian, $\exists\, x \in GL_n(k)$ s.t. $xGx^{-1} \subseteq$ diag. matrices.

$$\Rightarrow \forall g \in G, \ (xgx^{-1})_s = \operatorname{diag}(a_{11}, \dots, a_{nn}) \ \text{ if}$$

$$(xgx^{-1})_s = [a_{ij}].$$

Hence $G \longrightarrow G_s$, $g \mapsto g_s$ is a morphism of varieties

$\Rightarrow \quad G \longrightarrow G_s \times G_u$, $g \longmapsto (g_s, g_s^{-1} g)$ is a morphism of varieties. $\Rightarrow \phi$ is an isomorphism of algebraic groups. ∎

<u>Corollary</u>. G : commutative affine algebraic group.

G is connected $\Longleftrightarrow$ G_s and G_u are connected.

<u>Def.</u>. A linear algebraic group is called <u>diagonalizable</u> if it is isomorphic to a closed subgroup of $D_n := \{\operatorname{diag}(a_1, \dots, a_n) \mid a_i \in k^\times\}$ for some n.

• A linear algebraic group is called a torus if it is isomorphic to D_n for some n.

Theorem. Let G be an affine algebraic group. TFAE:

① G is commutative and consists of semisimple elements
(this is not needed)

② G is diagonalizable.

③ $X^*(G)$ is a f.g. abelian group and

$$k[G] = \bigoplus_{\chi \in X^*(G)} k\chi .$$

④ $\forall \rho : G \to GL(V)$ algebraic homomorphism, $V = \bigoplus_{\chi \in \Phi(G,V)} V_\chi$ for some finite subset $\Phi(G,V) \subseteq X^*(G)$.

Pf. ① ⇒ ②

$G = G_s \Rightarrow \exists x \in GL_n(k)$ s.t. $x G_s x^{-1} \subseteq D_n$

$\Rightarrow G_s$ is diagonalizable.

② ⇒ ③ $G \subseteq D_n$ as a closed subgroup.

$\Rightarrow k[G] \simeq k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] / I_G$.

Notice that $X^*(D_n)$ is a basis of $k[D_n] \simeq k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Since $X^*(D_n) \xrightarrow{\psi} X^*(G)$, $\chi \mapsto \chi|_G$ is a well-defined group homomorphism, $k[G]$ is spanned by $\operatorname{Im} \psi \subseteq X^*(G)$.

Since $X^*(G)$ are linearly independent and subset of the span of $\operatorname{Im}\Psi$, we get that $\underline{\operatorname{Im}\Psi = X^*(G)}$ and $X^*(G)$ is a basis of $k[G]$.

(Here we proved also that, if G is a closed subgroup of D_n, then $X^*(D_n) \longrightarrow X^*(G)$ is onto. Hence we get

$$\chi \longmapsto \chi|_G$$

<u>Lemma</u>. If $G_1 \subseteq G_2$ and G_2 is diagonalizable, then

$$X^*(G_2) \longrightarrow X^*(G_1), \quad \chi \longmapsto \chi|_{G_1}$$

is an onto group homomorphism.)

③ $\Rightarrow$ ④ Fix a basis for V and write $\rho: G \longrightarrow GL_n(k)$.

So $g \longmapsto \rho(g)_{ij}$ is in $k[G] \Rightarrow$

$\forall \chi \in X^*(G)$, $\exists\, a_\chi \in M_n(k)$ s.t.

$$\rho(g) = \sum_{\chi \in X^*(G)} a_\chi \, \chi(g).$$

So $\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$ implies

$$\sum_{\chi_1, \chi_2 \in X^*(G)} a_{\chi_1} a_{\chi_2} \, \chi_1(g_1)\, \chi_2(g_2) = \sum_{\chi} a_\chi \, \chi(g_1)\, \chi(g_2)$$

Hence by indepen. of elements of $X^*(G \times G)$ we get

- $\chi_1 \neq \chi_2 \Rightarrow a_{\chi_1} a_{\chi_2} = 0$

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- $\chi_1 \neq \chi_2 \Rightarrow a_{\chi_1} a_{\chi_2} = 0$
- $a_\chi^2 = a_\chi$
- $\rho(e) = I \Rightarrow \sum a_\chi = \mathbf{I}$

Let $V_\chi := a_\chi(V)$. Then

- $\forall v \in V \Rightarrow v = \sum a_\chi(v)$; $a_\chi(v) \in V_\chi$
- $\left.\begin{array}{l} \sum v_\chi = 0 \\ v_\chi \in V_\chi \end{array}\right\} \Rightarrow \forall \chi_0 \in X^*(G),\ a_{\chi_0}\left(\sum v_\chi\right) = 0$

$$\Rightarrow 0 = \sum a_{\chi_0} a_\chi v_\chi = a_{\chi_0}^2 v_{\chi_0} = v_{\chi_0}.$$

So $V = \bigoplus_{\chi \in \Phi} V_\chi$ for some finite subset $\Phi \subseteq X^*(G)$

and $\rho(g)\, v = \sum \chi(g)\, v_\chi$ where $v_\chi = a_\chi(v)$.

④ $\Rightarrow$ ② and ② $\Rightarrow$ ① are clear. ∎

Diag. alg. gp $\qquad\qquad$ Hopf algebra

$$\begin{tikzcd} G \arrow[rr, mapsto] \arrow[dr, mapsto] & & k[G] = \text{group algebra of } X^*(G) \\ & X^*(G) \arrow[ur, mapsto] & \end{tikzcd}$$

f.g. abelian group

- If $G \subseteq GL_n(k)$ and $\mathrm{char}(k) = p > 0$, then $X^*(G)$ has no p-torsion element (because of the Frob. automorphism of k.)
- Let M be a f.g. abelian group with no $\mathrm{char}(k)$-torsion element.

Let kM be the group algebra of M. Let

$$m^*: kM \longrightarrow kM \otimes kM, \quad m^* \delta_x := \delta_x \otimes \delta_x$$

$$i^*: kM \longrightarrow kM \quad , \quad i^* \delta_x = \delta_{-x}$$

$$e^*: kM \longrightarrow k \quad , \quad e^* \delta_x = 1 .$$

<u>Theorem</u> ① (kM, m^*, i^*, e^*) is a reduced commutative Hopf algebra of finite type; the corresponding affine algebraic group G_M is a diagonalizable affine group.

② $M \simeq X^*(G_M)$ canonically.

③ If G is diagonalizable affine group, then

$$G_{X^*(G)} \simeq G \quad \text{canonically}.$$

<u>Pf ① ②</u> $M \simeq \bigoplus \mathbb{Z}/_{q_i \mathbb{Z}} \Rightarrow kM \simeq k\, \mathbb{Z}/_{q_1 \mathbb{Z}} \otimes \cdots \otimes k\, \mathbb{Z}/_{q_m \mathbb{Z}}$.

$$k\, \mathbb{Z}/_{q\mathbb{Z}} \simeq k[T]/\langle T^q - 1 \rangle .$$

Since $p \nmid q$, $T^q - 1$ has q distinct roots. $(\gcd(T^q - 1, qT^{q-1}) = 1)$.

$$\Rightarrow k\, \mathbb{Z}/_{q\mathbb{Z}} \simeq \bigoplus_{i=0}^{q-1} k\, \zeta_q^i \qquad \text{is a reduced algebra}$$

$$1 + q\mathbb{Z} \longmapsto (1, \zeta_q, \ldots, \zeta_q^{q-1})$$

$\Rightarrow kM$ is a reduced algebra of finite-type.

One can easily check properties of Hopf algebra.

$\forall x \in M,\ g \in G_M := \mathrm{Hom}(kM, k)$, let $x(g) := g(\delta_x)$.

$$\Rightarrow \forall g_1, g_2 \in G_M, \quad x(g_1 \cdot g_2) = (g_1 \cdot g_2)(\delta_x)$$
$$= m \circ (g_1 \otimes g_2) \circ m^*(\delta_x)$$
$$= m \circ (g_1 \otimes g_2)(\delta_x \otimes \delta_x)$$
$$= m(g_1(\delta_x) \otimes g_2(\delta_x))$$
$$= x(g_1)\, x(g_2)$$

$$\left.\begin{array}{l} \Rightarrow M \hookrightarrow X^*(G_M) \subseteq k[G_M] = kM \\ \text{by the independ. of elements} \\ \text{of } X^*(G_M) \end{array}\right\} \Rightarrow M = X^*(G_M).$$

$\Downarrow$

G_M is diagonalizable.

③ We have seen that $k[G] = kX^*(G)$ and one can see that the Hopf algebra structure is as above $\Rightarrow$

$$k[G] \simeq k[G_{X^*(G)}] \text{ as Hopf algebras} \Rightarrow G \simeq G_{X^*(G)}. \quad \blacksquare$$

<u>Corollary</u>. $\mathrm{Hom}_{\text{alg. gps}}(G_1, G_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(X^*(G_2), X^*(G_1))$

if G_1 and G_2 are diagonalizable algebraic groups.

<u>Pf.</u>

$$\begin{tikzcd}
\mathrm{Hom}_{\text{alg. gps}}(G_1, G_2) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{\text{Hopf-alg.}}(k[G_2], k[G_1]) \arrow[d, equal] \\
\mathrm{Hom}(X^*(G_2), X^*(G_1)) \arrow[r, "\sim"] & \mathrm{Hom}(kX^*(G_2), kX^*(G_1))
\end{tikzcd}$$

$\Phi \longmapsto \Phi^*$

$\Phi \longmapsto \Phi^*|_{X^*(G_2)}$

$\mathbb{Z}$-mod

Hopf-alg

extending linearly

■

($G \mapsto X^*(G)$ fully faithful)

Corollary. Let G be a diagonalizable algebraic group.

① $G \simeq T \oplus F$ where T is a torus and F is a finite group and $\gcd(|F|, \mathrm{char}(k)) = 1$.

② G is a torus $\Leftrightarrow$ it is connected $\Leftrightarrow$ $X^*(G)$ is a free abelian gp.

Pf. $G \simeq G_{X^*(G)}$

$X^*(G) \simeq \mathbb{Z}^n \oplus M$ where $|M| < \infty$ with no p-torsion element

$$\Rightarrow G_{X^*(G)} \simeq G_{\mathbb{Z}^n} \times G_M \simeq \underbrace{D_n}_{\text{torus}} \times \underbrace{G_M}_{F}$$

$F \rightsquigarrow$ finite abelian with no p-element

Ex. complete the proof. ■

Theorem (Rigidity of diagonalizable groups)

G_1, G_2: diagonalizable; V : connected affine variety;

$\phi : V \times G_1 \to G_2$ a morphism s.t.

$\forall v \in V$, $\phi_v : G_1 \to G_2$, $\phi_v(g) := \phi(v, g)$ is an algebraic gp homom.

$\Rightarrow$ $\phi(v, g)$ is independ. of v.

(a group homomo. between two diag. alg. gps cannot be algebraically deformed.)

Pf. For any $\psi \in X^*(G_2)$,

$$(v, g) \longmapsto \psi(\phi(v,g))$$

is in $k[V \times G_1] \simeq k[V] \otimes k[G_1]$. So $\forall \chi \in X^*(G_1), \exists f_{\psi,\chi} \in k[V]$

s.t. $$\psi(\phi(v,g)) = \sum_{\chi \in X^*(G_1)} f_{\psi,\chi}(v)\, \chi(g)$$

So $\psi(\phi(v,g_1))\,\psi(\phi(v,g_2)) = \psi(\phi(v,g_1g_2))$ implies

$$\sum_{\chi_1,\chi_2} f_{\psi,\chi_1}(v)\, f_{\psi,\chi_2}(v)\, \chi_1(g_1)\, \chi_2(g_2)$$

$$= \sum_{\chi} f_{\psi,\chi}\, \chi(g_1)\, \chi(g_2).$$

Hence again by the indep. of elements of $X^*(G_1 \times G_1)$ we have

- $f_{\psi,\chi_1}(v)\, f_{\psi,\chi_2}(v) = 0$ if $\chi_1 \neq \chi_2$
- $\left.\begin{array}{l} f_{\psi,\chi}{}^2 = f_{\psi,\chi} \\ V: \text{irred.} \Rightarrow k[V] \text{ integral domain} \end{array}\right\} \Rightarrow f_{\psi,\chi} = 0$ or $f_{\psi,\chi} = 1$.

$\Rightarrow \psi(\phi(v,g))$ is <u>either</u> 0 <u>or</u> $\chi_\psi(g)$ for some $\chi_\psi \in X^*(G_1)$

In particular, $\psi(\phi(v,g))$ is indep. of v for any $\psi \in X^*(G_2)$

$\Rightarrow \phi(v,g)$ is indep. of v as $k[G_2]$ is gener. by $X^*(G_2)$. ∎

<u>Corollary</u>. Suppose H is a diagonalizable subgp of an affine alg. gp G;

then $Z_G(H)^\circ = N_G(H)^\circ$ and $N_G(H)/Z_G(H)$ is finite.

<u>Pf.</u> Let $V := N_G(H)^\circ$ and $\phi : V \times H \longrightarrow H$,

$$\phi(g,h) := ghg^{-1}.$$

$\Rightarrow$ by rigidity of H, $\phi(g,h) = \phi(e,h) = h$

$\Rightarrow$ $g \in Z_G(H)$ $\Rightarrow$ $N_G(H)^\circ \subseteq Z_G(H) \subseteq N_G(H)$

$\Rightarrow$ $Z_G(H)^\circ = N_G(H)^\circ$ and $N_G(H)/N_G(H)^\circ \twoheadrightarrow N_G(H)/Z_G(H)$

and so $N_G(H)/Z_G(H)$ is finite. $\blacksquare$