Background on AG: differential of a morphism

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Here we briefly review the definition of differential of a morphism $\phi : X \to Y$ where X and Y are affine varieties. We will see how $d\phi|_x$ acts on $T_x X$ under various definitions of $T_x X$:

<u>1st definition</u>. $T_x X$ is the "fiber" over x of

$$X(k[\varepsilon]) \longrightarrow X(k).$$

$$d\phi|_x (v) = w \quad \text{if} \quad \phi(x+\varepsilon v) = \phi(x) + \varepsilon w.$$

Notice that ϕ induces a map and the following diagram is commutative.

$$\begin{tikzcd} X(k[\varepsilon]) \arrow[r, "\phi_{k[\varepsilon]}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y(k[\varepsilon]) \arrow[d] \\ X(k) \arrow[r, "\phi"] & Y(k) \end{tikzcd}$$

So $\quad v \in T_x X \implies x + \varepsilon v \in X(k[\varepsilon])$

$\implies \phi(x+\varepsilon v) \in Y(k[\varepsilon])$

and <u>mod ε</u> it is mapped to $\phi(x)$

$\implies \phi(x+\varepsilon v) = \phi(x) + \varepsilon w \in Y(k[\varepsilon])$

and $w \in T_{\phi(x)} Y$.

<u>Compatibility with calculus</u>!

Suppose $X \subseteq k^n$ and $Y \subseteq k^m$, and $\phi(x_1,\dots,x_n) = (f_1(\vec{x}),\dots,f_m(\vec{x}))$.

Then by Taylor expansion of the polynomials f_i's at $\vec{x}$ we have

$f_i(x+\varepsilon v) = f_i(x) + \varepsilon \, df_i\big|_x (v)$. So

$$\phi(x+\varepsilon v) = \phi(x) + \varepsilon \begin{bmatrix} \nabla f_1(x) \\ \vdots \\ \nabla f_m(x) \end{bmatrix} v .$$

Hence $d\phi\big|_x : T_x X \longrightarrow T_{\phi(x)} Y$ with the given embeddings of $T_x X \subseteq k^n$ and $T_{\phi(x)} Y \subseteq k^m$ can be given as the restriction of the linear map given by $J_{f_i} := \begin{bmatrix} \partial_1 f_1(x) & \cdots & \partial_n f_1(x) \\ \vdots & & \vdots \\ \partial_1 f_m(x) & \cdots & \partial_n f_m(x) \end{bmatrix}$.

[Notice that J_{f_i} depends on the choice of f_i's among the functions with the same rest. to X, but $J_{f_i}\big|_{T_x X}$ just depends on $f_i\big|_X$.]

So $d\phi\big|_x$ is a linear map from $T_x X$ to $T_{\phi(x)} X$.

<u>2^{nd} definition</u>. $T_x X = \mathrm{Der}_k(k[X], k_x)$.

Let $\phi : X \longrightarrow Y$ be a morphism. So we get the following diagram:

$$\begin{tikzcd} k[Y] \arrow[r, "\phi^*"] \arrow[d, dashed, "d\phi|_x(\delta)"'] & k[X] \arrow[d, "\delta"] \\ k_{\phi(x)} \arrow[r, "="] & k_x \end{tikzcd}$$

- $k[Y] \xrightarrow{\phi^*} k[X]$ makes $k = k_x$ a $k[Y]$-mod:

$$\forall\ f\in k[Y],\ c\in k_x,\quad f\bullet c = \phi^*(f)\bullet c$$
$$= \phi^*(f)(x)\ c$$
$$= f(\phi(x))\ c,$$

which is $k_{\phi(x)}$.

- We have already seen the ϕ^* induces a k-linear map

$$\mathrm{Der}_k(k[X], k_x) \longrightarrow \mathrm{Der}_k(k[Y], k_{\phi(x)})$$
$$\delta \longmapsto \delta\circ\phi^* =: d\phi|_x(\delta).$$

<u>Compatibility with the 1^{st} def.</u>

For $x\in X(k)$;
$$\underset{X(k[\varepsilon])}{\overset{x+\varepsilon v}{\cap}} \longmapsto \underset{\mathrm{Der}_k(k[X],k_x)}{\overset{\delta_v}{\cap}},$$
where $\delta_v([f]) = df|_x(v)$.

Inverse of this map is :

$$\underset{\mathrm{Der}_k(k[X],k_x)}{\overset{\delta}{\cap}} \longmapsto \underset{X(k[\varepsilon])}{\overset{x+\varepsilon v_\delta}{\cap}},$$

where $\quad \forall f\in k[X],\ f(x+\varepsilon v_\delta) = f(x)+\varepsilon\,\delta(f)$.

We have to show $\phi(x+\varepsilon v_\delta) = \phi(x)+\varepsilon v_{\delta\circ\phi^*}$

$\phi(x+\varepsilon v_\delta) = \phi(x)+\varepsilon v_{\delta\circ\phi^*} \iff$

$\forall f\in k[Y],\ f(\phi(x+\varepsilon v_\delta)) = f(\phi(x)+\varepsilon v_{\delta\circ\phi^*}) \iff$

$$\phi^*(f)(x+\varepsilon v_\delta) = f(\phi(x)) + \varepsilon\,(\delta\circ\phi^*)(f) \iff$$

$$\phi^*(f)(x) + \varepsilon\,\delta(\phi^*(f)) = f(\phi(x)) + \varepsilon\,(\delta\circ\phi^*)(f) \quad \checkmark$$

<u>3rd definition.</u> $\quad T_x X = \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*.$

Let $\phi: X \to Y$ be a morphism;

For $\ell \in \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*$ and $f \in \mathfrak{m}_{\phi(x)}$, let

$$\left(d\phi|_x(\ell)\right)(f + \mathfrak{m}_{\phi(x)}^2) := \ell(\phi^*(f) + \mathfrak{m}_x^2)$$

<u>Well-definedness</u>. $\bullet$ $f \in \mathfrak{m}_{\phi(x)} \iff f(\phi(x)) = 0$

$$\iff \phi^*(f)(x) = 0$$

$$\iff \phi^*(f) \in \mathfrak{m}_x.$$

$\Rightarrow \phi^*(\mathfrak{m}_{\phi(x)}^2) \subseteq \mathfrak{m}_x^2 \rightsquigarrow$ indepen. of the choice of f.

And it is linear in $f + \mathfrak{m}_{\phi(x)}^2$. So it is well-defined

map $\quad d\phi|_x : \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^* \longrightarrow \left(\mathfrak{m}_{\phi(x)}/\mathfrak{m}_{\phi(x)}^2\right)^*.$

<u>Compatibility with the 2nd def.</u>

$\bullet$ $$\underset{\mathrm{Der}_k(k[X], k_x)}{\overset{\delta}{\cap}} \longmapsto \underset{\left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*}{\overset{\ell_\delta}{\cap}}, \quad \text{where}$$

$$\ell_\delta(f + \mathfrak{m}_x^2) := \delta(f).$$

$$\ell_\delta(f + \mathfrak{m}_x^2) := \delta(f) .$$

$$\cdot \quad \begin{array}{ccc} \ell & \longmapsto & \delta_\ell \\ \cap & & \cap \\ (\mathfrak{m}_x/\mathfrak{m}_x^2)^* & & \mathrm{Der}_k(k[X], k_x) \end{array} \quad , \text{ where}$$

$$\delta_\ell(f) := \ell(f - f(x) + \mathfrak{m}_x^2) .$$

For a morphism $\phi : X \to Y$, we have to show

$$\begin{array}{ccc} \ell_{\delta\circ\phi^*}(f + \mathfrak{m}_{\phi(x)}^2) & \overset{?}{=} & \ell_\delta(\phi^*(f) + \mathfrak{m}_x^2) \\ \| & & \| \\ (\delta\circ\phi^*)(f) & & \delta(\phi^*(f)) , \end{array}$$

which is clear.

<u>4^{th} definition.</u> $\quad T_x X = \mathrm{Hom}_{k[X]}(\Omega_{k[X]/k}, k_x)$

Let $\phi : X \to Y$ be a morphism. Then $\phi^* : k[Y] \to k[X]$ induces a $k[Y]$-mod. homomorphism $\phi^{*\circ} : \Omega_{k[Y]/k} \to \Omega_{k[X]/k}$ where $\phi^{*\circ}(d_{k[Y]/k} f) = d_{k[X]/k}\, \phi^*(f)$.

Then

$$\begin{tikzcd} \Omega_{k[Y]/k} \arrow[r, "\phi^{*\circ}"] \arrow[d, "d\phi|_x(\alpha)"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega_{k[X]/k} \arrow[d, "\alpha"] \\ k_{\phi(x)} \arrow[r, "="] & k_x \end{tikzcd}$$

So $d\phi|_x(\alpha) = \alpha \circ \phi^{*\circ}$.

<u>Compatibility with the 2^{nd} def.</u>

Compatibility with the 2nd def.

$$\begin{array}{ccc} \alpha & \longmapsto & \delta_\alpha \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathrm{Hom}_{k[X]}(\Omega_{k[X]/k}, k_x) & & \mathrm{Der}_k(k[X], k_x) \end{array}$$

where $\delta_\alpha = \alpha \circ d_{k[X]/k}$

Let $\phi : X \to Y$ be a morphism. We have to show

$$\delta_\alpha \circ \phi^* = \delta_{\alpha \circ \phi^{*\circ}} \qquad (\text{why ?})$$

$$\begin{aligned} \delta_\alpha \circ \phi^* &= (\alpha \circ d_{k[X]/k}) \circ \phi^* \\ &= \alpha \circ (d_{k[X]/k} \circ \phi^*) \\ &= \alpha \circ \phi^{*\circ} \circ d_{k[Y]/k} = \delta_{\alpha \circ \phi^{*\circ}} . \end{aligned}$$

5th definition $\quad T_x X = \Omega_{k[X]/k}(x)^* := \mathrm{Hom}_{k_x}(\Omega_{k[X]/k} \otimes k_x, k_x)$

Let $\phi : X \to Y$ be a morphism. Then $\phi^* : k[Y] \to k[X]$

induces $\quad \Omega_{k[Y]/k} \otimes k_{\phi(x)} \longrightarrow \Omega_{k[X]/k} \otimes k_x$

$$\omega \otimes 1 \longmapsto \phi^{*\circ}(\omega) \otimes 1 .$$

Let $d\phi|_x$ be the dual of $\phi^{*\circ} \otimes \mathrm{id.}$, i.e.

$\forall\, \ell \in \Omega^*_{k[X]/k}$, $\forall\, \omega \in \Omega_{k[Y]/k}$,

$$(d\phi|_x(\ell))(\omega \otimes 1) := \ell\,(\phi^{*\circ}(\omega) \otimes 1) .$$

Compatibility with the 4^{th} def.

$$\begin{array}{ccc} \alpha & \longmapsto & \bar{\alpha} \\ \cap & & \cap \\ \mathrm{Hom}_{k[X]}(\Omega_{k[X]/k}, k_x) & & \mathrm{Hom}_{k_x}(\Omega_{k[X]/k}(x), k_x) \end{array}$$

where $\hat{\alpha}(\omega \otimes 1) := \alpha(\omega)$.

It is an isomorphism (why?)

[$\mathrm{Hom}_{k_x}(\Omega_{k[X]/k} \otimes k_x, k_x) \simeq \mathrm{Hom}_{k[X]}(\Omega_{k[X]/k}, k_x)$

$$\mathrm{Hom}_S(M_1 \otimes_R M_2, N) \simeq \mathrm{Hom}_R(M_1, \mathrm{Hom}_S(M_2, N))$$

$$\Longrightarrow \mathrm{Hom}_{k_x}(\Omega_{k[X]/k} \otimes_k k_x, k_x) \simeq \mathrm{Hom}_{k[X]}(\Omega_{k[X]/k}, k_x).$$]

Now compatibility is clear.