Background on AG: Tangent space

Thursday, May 4, 2017 9:46 AM

Suppose $f_i \in \mathbb{R}[x_1, \dots, x_m]$ and X is the set of common zeros of f_i's over $\mathbb{R}$. To find the tangent plane of X at x_0, from calculus we know

$$T_{x_0}X = \left\{ \vec{r}\,'(0) \;\middle|\; \begin{array}{l} \cdot\ r \text{ is diff. curve in } X \\ \cdot\ r(0) = x_0 \end{array} \right\}$$

Since $\mathrm{Im}(r) \subseteq X$, for any i, we have

$$(f_i \circ r)(t) = 0.$$

And so by chain rule $\nabla f_i(x_0) \cdot \vec{r}\,'(0) = df_i|_{x_0}(\vec{r}\,'(0))$.

In calculus, assuming $T_{x_0}X$ exists, we use the above technique and write $T_{x_0}X = \bigcap \ker df_i|_{x_0}$.

- Let's make an observation:

$$\vec{v} \in \ker df|_{x_0} \Leftrightarrow f(x_0 + \varepsilon v) = 0 \quad \text{in } \mathbb{R}[\varepsilon] = \mathbb{R} \oplus \mathbb{R}\varepsilon$$

where $\varepsilon^2 = 0$.

(why: f is a polynomial. So it has a (formal) Taylor expansion at x_0:

$$f(x_0 + y) = f(x_0) + df|_{x_0}(y) + \text{higher degree in terms of } y_i\text{'s}.$$

$$\Rightarrow f(x_0 + \varepsilon v) = f(x_0) + \varepsilon\, df|\,(v)$$

$$= \varepsilon\, df|_{x_0}(v).\ )$$

$$= \ldots |_{x_0}(\ldots)$$

So $\bigcap \ker df_i|_{x_0} = V(df_i|_{x_0})$

$= $ fiber over x_0 of

$V(\{f_i\})(\mathbb{R}[\varepsilon]) \longrightarrow V(\{f_i\})(\mathbb{R})$.

- <u>1^{st} definition of tangent space</u>

Let $\mathfrak{a}$ be an ideal of $k[\underline{x}_1, \cdots, \underline{x}_n]$. Then the tangent bundle of $V(\mathfrak{a})(k)$ defined to be $V(\mathfrak{a})(k[\varepsilon])$, and the tangent space of $V(\mathfrak{a})(k)$ at x is the fiber over x of $V(\mathfrak{a})(k[\varepsilon]) \longrightarrow V(\mathfrak{a})(k)$.

(This definition can be easily extended to schemes over an arbitrary ring k.)

Let's go back to calculus. For a given $v \in T_{x_0}X$ and a (nice) function $f : X \longrightarrow \mathbb{R}$, we can say what the rate of change of f is in the direction of v:

Suppose r is a curve in X s.t. $r(0) = x_0$ and $r'(0) = v$. Then the rate of change of f in the direction of v is

$(f \circ r)'(0) = df|_{x_0}(v)$ (as you can see it is indep. of choice of r.)

So we get $\delta_{v,x_0} : \mathbb{R}[X] \longrightarrow \mathbb{R}$,

$$\delta_{v,x_0}(f) := df|_{x_0}(v).$$

Fixing v and varying x_0, we get $D_v : \mathbb{R}[X] \longrightarrow \mathbb{R}[X]$,

$$D_v(f)(x) := \delta_{v,x}(f).$$

Basic properties of $\delta_{v,x}$ and D_v.

$$\begin{cases} \bullet\ \delta_{v,x} \text{ is linear} \\ \bullet\ \delta_{v,x_0}(f_1 f_2) = f_1(x_0)\, df_2|_{x_0}(v) + f_2(x_0)\, df_1|_{x_0}(v) \\ \qquad\qquad\quad = f_1(x_0)\, \delta_{v,x_0}(f_2) + f_2(x_0)\, \delta_{v,x_0}(f_1). \end{cases}$$

$$\begin{cases} \bullet\ D_v \text{ is linear} \\ \bullet\ D_v(f_1 f_2) = f_1\, D_v(f_2) + f_2\, D_v(f_1). \end{cases}$$

Definition. Let A be a k-algebra, and M be a (left) A-mod.

$D: A \to M$ is called a k-derivation if

① D is k-linear.

② $D(a_1 a_2) = a_1 \cdot D(a_2) + a_2 \cdot D(a_1)$.

. Viewing $\mathbb{R}$ as an $\mathbb{R}[X]$-mod via $f \cdot \alpha := f(x_0)\alpha$, i.e.

$\mathbb{R} \simeq \mathbb{R}[X]/\mathfrak{m}_{x_0}$, we see that

$$\delta_{v,x} : \mathbb{R}[X] \to \mathbb{R} \quad \text{and} \quad D_v : \mathbb{R}[X] \to \mathbb{R}[X]$$

are $\mathbb{R}$-derivations.

<u>2^{nd} definition of tangent space</u>

For an algebraic set X, the tangent space $T_x X$ of X at x is

$$\mathrm{Der}_k(k[X], k_x) := \{ \delta : k[X] \to k_x \mid \delta : k\text{-derivation} \}$$

where $k_x = k[X]/\mathfrak{m}_x$, i.e. $f \cdot \alpha := f(x)\,\alpha \quad \forall f \in k[X]$ and $\alpha \in k$.

<u>Lemma</u>. (<u>3^{rd} def.</u>) $\lambda : \mathrm{Der}_k(k[X], k_x) \xrightarrow{\sim} \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^* := \mathrm{Hom}_{k_x}\left(\mathfrak{m}_x/\mathfrak{m}_x^2, k\right)$

$$\lambda(\delta)([f]) := \delta(f)$$

<u>Pf.</u> <u>Well-defined</u>. $f_1, f_2 \in \mathfrak{m}_x \Rightarrow \delta(f_1 f_2) = \underbrace{f_1(x)\,\delta(f_2) + f_2(x)\,\delta(f_1)}_{=0}$

$\Rightarrow \delta(\mathfrak{m}_x^2) = 0$. So $[f_1] = [f_2]$ implies $\delta(f_1) = \delta(f_2)$.

<u>linear map</u> is clear.

<u>Injective</u>. $\lambda(\delta) = 0 \Rightarrow \forall f \in k[X],\ \delta(f) = 0 \Rightarrow \delta = 0$.

<u>Surjective</u>. $\ell : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$ linear.

Let $\delta(f) := \ell([f - f(x)])$. Then

$$\delta(f_1 f_2) = \ell([f_1 f_2 - f_1(x) f_2(x)])$$

$$= \ell\Big(\Big[\underbrace{(f_1 - f_1(x))(f_2 - f_2(x))}_{\text{in } \mathfrak{m}_x^2}$$

$$+ (f_1 - f_1(x))\, f_2(x)$$

$$+ f_1(x)\,(f_2 - f_2(x))]\,)$$

$$= f_2(x)\, \ell([f_1 - f_1(x)]) + f_1(x)\, \ell([f_2 - f_2(x)])$$

$$= f_2(x)\, \delta(f_1) + f_1(x)\, \delta(f_2). \qquad \blacksquare$$

<u>Lemma</u>. ($1^{st} \equiv 2^{nd}$)

$$V(k[\varepsilon]) \longrightarrow \coprod_{x \in X} \mathrm{Der}(k[X], k_x)$$
$$x + \varepsilon v \longmapsto \delta_{x,v}$$

where $X = V(k)$ and $\delta_{x,v}(f) := df|_x(v)$.

and, it induces isomorphism of linear spaces

$$\underbrace{\left(\text{fiber over } x_0 \text{ of } V(k[\varepsilon]) \longrightarrow V(k)\right)}_{\circledast} \xrightarrow{\sim} \mathrm{Der}(k[X], k_x).$$

<u>Pf.</u> It is enough to prove the second assertion.

<u>Well-defined</u>. $f \in I(X) \Rightarrow f(x+\varepsilon v) = f(x) + \varepsilon\, df|_x(v) = 0 \Rightarrow df|_x(v) = 0$.

$\Rightarrow df|_{x_0}$ is a well-defined function on $\circledast$.

. It is easy to check that $\delta_{x_0,v} \in \mathrm{Der}(k[X], k_{x_0})$ and $v \longmapsto \delta_{x_0,v}$ is k-linear.

<u>Injective</u>. $\delta_{x_0,v} = 0 \Rightarrow \forall f \in k[X],\ df|_{x_0}(v) = 0 \xrightarrow{\text{Coord. functions}} v = 0$.

<u>Surjective</u>. $\delta \in \mathrm{Der}(k[X], k_{x_0})$, let $v := (\delta(\underline{x_i}))$ where $\underline{x_i}$'s are the coordinate functions. Then $\delta_{x_0,v}(\underline{x_i}) = \delta(\underline{x_i})$. So $\delta_{x_0,v} = \delta$. $\blacksquare$

In order to extend this to all the varieties, we need to get a <u>local treatment</u> of the same ideal: working with $\mathcal{O}_x$ instead of $k[X]$.

<u>Lemma</u>. $A \xrightarrow{\phi} B$ a k-alg. homomorphism

M: B-module (and so A-mod via ϕ)

$$\Rightarrow \quad 0 \to \mathrm{Der}_A(B,M) \to \mathrm{Der}_k(B,M) \to \mathrm{Der}_k(A,M)$$
$$\delta \mapsto \delta$$
$$\delta \mapsto \delta \circ \phi$$

is an exact sequence.

<u>Pf.</u> • $\forall a_1, a_2 \in A$, $\delta(\phi(a_1 a_2)) = \delta(\phi(a_1)\phi(a_2))$

$$= \phi(a_1) \cdot \delta(\phi(a_2)) + \phi(a_2) \cdot \delta(\phi(a_1))$$

So $\delta \circ \phi \in \mathrm{Der}_k(A,M)$.

• $\delta(\phi(a)) = 0 \ \forall a \in A \Rightarrow \delta(\phi(a)\, b) = \phi(a)\, \delta(b)$

$\Rightarrow \delta$ is A-linear $\Rightarrow \delta \in \mathrm{Der}_A(B,M)$. ∎

<u>Corollary</u>. $\mathrm{Der}_k(\mathcal{O}_x, k_x) \simeq \mathrm{Der}_k(k[X], k_x)$.

<u>Pf.</u> $k[X] \to k[X]_{\mathfrak{m}_x} \simeq \mathcal{O}_x$ induces

$$0 \to \mathrm{Der}_{k[X]}(\mathcal{O}_x, k_x) \to \mathrm{Der}_k(\mathcal{O}_x, k_x) \to \mathrm{Der}_k(k[X], k_x).$$

• $\delta \in \mathrm{Der}_{k[X]}(\mathcal{O}_x, k_x) \Rightarrow g(x)\, \delta(f/g) = (g \cdot \delta)(f/g)$

$$= \delta(g \cdot f/g) = \delta(f)$$
$$= (f\cdot\delta)(1) = f(x)\,\delta(1) = 0.$$
$$\Rightarrow \delta(f/g) = 0. \qquad (\text{injective})$$

- $\bar{\delta} \in \mathrm{Der}_k(k[X], k_x)$, let $\delta(f/g) := \dfrac{g(x)\,\bar{\delta}(f) - f(x)\,\bar{\delta}(g)}{g(x)^2}$.

Then $\delta(f/1) = \bar{\delta}(f)$. And δ is a derivation (why?).

(Surjective). ■

<u>Lemma</u>. If U and V are open affine nbhds of x in a variety X and $V \subseteq U$, then

$$T_x V \simeq T_x U$$

canonically.

<u>Pf</u>. The injection maps induce isomorphisms $\mathcal{O}_{X,x} \simeq \mathcal{O}_{U,x} \simeq \mathcal{O}_{V,x}$

And so we get isomorphisms of

$$\mathrm{Der}_k(\mathcal{O}_{X,x}, k_x) \simeq \mathrm{Der}_k(\mathcal{O}_{U,x}, k_x) \simeq \mathrm{Der}_k(\mathcal{O}_{V,x}, k_x).$$

□

<u>Def</u>. $T_x X := \varprojlim_{\substack{U:\text{open}\\ \text{affine}\\ \text{nbhd of } x}} \mathrm{Der}_k(\mathcal{O}_{U,x}, k_x)$ (formal definition)

$$= \mathrm{Der}_k\Big(\underbrace{\varinjlim \mathcal{O}_{U,x}}_{\simeq \cdots}, k_x\Big).$$

$\mathcal{O}_{X,x}$

Def. (differential of a morphism)

$\phi : X \to Y$ a morphism of algebraic sets.

Then $\phi^* : k[Y] \to k[X]$ induces a map

$$d\phi_x : \underset{\displaystyle T_x X}{\underset{\shortparallel}{\mathrm{Der}_k(k[X], k_x)}} \longrightarrow \underset{\displaystyle T_{\phi(x)} Y}{\underset{\shortparallel}{\mathrm{Der}_k(k[Y], k_{\phi(x)})}}$$

which is called the differential of ϕ at x.

We can extend this definition to morphisms of varieties.

Definition. We say x is a <u>simple point</u> of X or

X is <u>smooth</u> in x or

X is non-singular in x if

$$\dim T_x X = \dim X.$$

To understand <u>simple points</u> we study <u>module of differentials</u> $\Omega_{A/R}$