Background on AG: dimension of algebraic sets

Tuesday, April 18, 2017 11:52 AM

Def. Let $X \subseteq k^n$ be an algebraic set. Then $\dim X := \dim k[X]$ where $\dim k[X]$ is the Krull dimension of $k[X]$. (Recall from commutative algebra:

$$\dim A := \sup\{n \mid \exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Spec}(A)\}.)$$

So geometrically it says:

$$\dim X := \sup\{n \mid \exists\, X \supseteq \underbrace{X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n}_{\text{closed subsets.}} \text{ irreducible}\}.$$

- The following theorems from commutative algebra help us show dim is a well-behaved notion. Some of these statements might be not proved in Math 200C.

Thm. $A \subseteq B$ is an integral extension $\Rightarrow$

- $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ is onto with finite fibers
- $\dim A = \dim B$.

Thm (Noether normalization)

A f.g. k-algebra, integral domain $\Rightarrow$

$\exists\, x_1, \ldots, x_n \in A$ which are algebraically indep. and

$k[x_1, \ldots, x_n] \subseteq A$ is integral.

Thm. $\dim k[x_1, \ldots, x_n] = n$

- And so for any f.g. k-alg, integral domain, A

$$\dim A = \operatorname{tr.deg}_k Q(A).$$

<u>Proposition</u> $X \subsetneq Y$ irreducible algebraic sets $\Rightarrow \dim X < \dim Y$.

<u>Pf.</u> By the above theorem, $\dim X = d < \infty$. Let $X_0 \subsetneq \cdots \subsetneq X_d = X$ be a chain of irreducible, closed subspaces of X. Then

$$X_0 \subsetneq \cdots \subsetneq X_d = X \subsetneq Y$$

is a chain of irreducible, closed subspaces of Y of length $d+1$.

So $\dim Y \geq d+1$. ∎

[<u>Thm</u>. ① $ht(\langle f_1, \ldots, f_m \rangle) \leq m$ in any Noeth. ring.

② If $ht(\langle f_1, \ldots, f_m \rangle) = m$, then $\forall$ minimal prime $\mathfrak{p}$ of $\langle f_1, \ldots, f_m \rangle$ we have $ht(\mathfrak{p}) = m$.

(Complete intersection varieties.)

③ $\mathfrak{p} \subseteq \mathfrak{q} \in \operatorname{Spec}(k[x_1, \ldots, x_n]) \Rightarrow$ all the saturated chains have the same length.]