First steps towards defining the quotient space G/H

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Let G be an affine algebraic group. Let H be a closed subgroup of G. We would like to view G/H as a variety where $G \to G/H$ is a morphism and $G \curvearrowright G/H$ is an action of an algebraic group on a variety (and some infinitesimal conditions), and we'd like to get a universality property.

Lemma. $\{g \in G \mid s(g)(I(H)) = I(H)\} = H$.

Pf. • $s(g)\, I(H) = I(H) \implies \forall f \in I(H),\ s(g)(f)(e) = 0$

$$\Rightarrow \forall f \in I(H),\ f(g^{-1}) = 0$$

$$\Rightarrow g^{-1} \in H \Rightarrow g \in H.$$

• $\forall h \in H, \forall f \in I(H),\ \forall h' \in H,\ (s(h)(f))(h')$

$$= f(\underbrace{h^{-1}h'}_{\in H})$$

$$= 0 \quad \text{as } f \in I(H)$$

$\Rightarrow s(h)(f) \in I(H)$. $\blacksquare$

Lemma. For any closed subgp H of G,

$\exists\ \rho : G \to GL(V)$: an algebraic group homomorphism,

$\exists\ W \subseteq V$ s.t.

V and W are finite-dimensional k-spaces

And $H = \{ g \in G \mid \rho(g)(W) = W \}$.

<u>Pf.</u> Suppose $I(H)$ is generated by $f_1, \dots, f_m$ as an ideal. Since $H \curvearrowright G$, after enlarging the list of f_i's, if needed, we can assume that $W := \operatorname{span}\{f_1, \dots, f_m\}$ is H-invariant. Let V be the G-submod. of $k[G]$ which is generated by W. So $\dim V < \infty$.

. $W \subseteq V$ are finite-dimensional, and we have an algebraic group homomorphism $G \xrightarrow{\rho} GL(V)$ (why?) s.t. $H \subseteq \{ g \in G \mid \rho(g)(W) = W \}$.

. $\rho(g)(W) = W \Rightarrow s(g)(\underbrace{\langle W \rangle}_{\text{ideal generated by } W}) = \langle W \rangle$

as $s(g) : k[G] \xrightarrow{\sim} k[G]$ and $s(g)\, W = W$.

Hence $s(g)\, I(H) = I(H)$, which implies $g \in H$. ■

S <u>Proposition.</u> For any closed subgp H of an affine algebraic group G,

$\exists\ \rho : G \to GL(V)$: an affine algebraic group homomorphism

$\exists\ \ell \subseteq V$ a line s.t.

$\dim V < \infty$ and $\{ g \in G \mid \rho(g)(\ell) = \ell \} = H$.

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· To prove this proposition, I need to recall the exterior algebra $\Lambda(V)$ of a vector space V:

$$\Lambda(V) = T(V)/_{I} \quad \text{where } T(V) := k \oplus V \oplus V\otimes V \oplus \cdots$$

is the tensor algebra and $I = \langle x\otimes x \mid x \in V\rangle$.

$\Lambda(V)$ has a natural grading. Its k^{th} grade is denoted by $\Lambda^k V$ which is isomorphic to $\otimes^k V/_{I\cap \otimes^k V}$. For $v_1, \dots, v_k \in V$, let $v_1\wedge\cdots\wedge v_k := v_1\otimes\cdots\otimes v_k + (I\cap \otimes^k V)$. So

$$\underbrace{(v_1+v_2)\wedge(v_1+v_2)}_{0} = \underbrace{v_1\wedge v_1}_{0} + v_1\wedge v_2 + v_2\wedge v_1 + \underbrace{v_2\wedge v_2}_{0}$$

$$\Rightarrow v_1\wedge v_2 = - v_2\wedge v_1$$

$$\Rightarrow \boxed{v_{\sigma_1}\wedge\cdots\wedge v_{\sigma_n} = \operatorname{sgn}(\sigma)\, v_1\wedge\cdots\wedge v_n \quad \text{for any } \sigma\in S_n.}$$

· For $a_{ij}\in k$, $\left(\sum_{j=1}^{m} a_{1j}\, v_j\right)\wedge\left(\sum_{j=1}^{m} a_{2j}\, v_j\right)\wedge\cdots\wedge\left(\sum_{j=1}^{m} a_{nj}\, v_j\right)$

$$= \sum_{1\le j_1,\dots,j_n\le m} (a_{1j_1} a_{2j_2}\cdots a_{nj_n})\, v_{j_1}\wedge\cdots\wedge v_{j_n}$$

[So, if $n>m$, then $v_{j_1}\wedge\cdots\wedge v_{j_n} = 0$; and so

$\Lambda^n V = 0$ if $\dim V < n$.]

$$= \sum_{1\le i_1<\cdots<i_n\le m} \left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j\, i_{\sigma(j)}}\right) v_{i_1}\wedge\cdots\wedge v_{i_n}. \quad \circledast$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nm} \end{bmatrix}_{n\times m}$ and, for any $I = \{i_1 < \cdots < i_n\}$, let

$$A_I := \begin{bmatrix} a_{1\, i_1} & \cdots & a_{1\, i_n} \\ \vdots & \ddots & \vdots \\ a_{n\, i_1} & \cdots & a_{n\, i_n} \end{bmatrix}_{n\times n}.$$

Then $\circledast$ implies

$$\boxed{\left(\sum_{j=1}^{m} a_{1j} v_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^{m} a_{nj} v_j\right) = \sum_{\substack{I \subseteq [1..m] \\ |I| = n}} \det(A_I)\, v_{i_1} \wedge \cdots \wedge v_{i_n}.}$$

And so, if $V = \operatorname{span}(v_1, \ldots, v_m)$, then

$$\bigwedge^n V = \operatorname{span}(v_{i_1} \wedge \cdots \wedge v_{i_n} \mid 1 \le i_1 < \cdots < i_n \le m).$$

Suppose $e_1, \ldots, e_m$ are linearly independent. For any $I = \{i_1 < \cdots < i_n\}$, let $e_I := e_{i_1} \wedge \cdots \wedge e_{i_n}$.

If $I_1, \ldots, I_l$ are distinct subsets of size n of $[1..m]$, then $e_{I_1}, \ldots, e_{I_l}$ are linearly independent (why?).

[Pf. Suppose $\sum c_i e_{I_i} = 0 \Rightarrow \sum_i c_i e_{I_i} \wedge e_{I_j^c} = 0$

where $I_j^c := [1..m] \setminus I_j$.

Since, $\forall i \ne j$, $I_i \cap I_j^c \ne \emptyset$, $e_{I_i} \wedge e_{I_j^c} = 0$. Hence $c_j\, e_1 \wedge \cdots \wedge e_m = 0$. So it is enough to show

$e_1 \wedge \cdots \wedge e_m \ne 0$ if $e_1, \ldots, e_m$ are linearly independent, which is clear (why?).]

<u>Proposition.</u> If $e_1, \dots, e_m$ is a basis of V, then

$$\{ e_{i_1} \wedge \cdots \wedge e_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq m \} \text{ is a basis of } \Lambda^n V.$$

In particular, $\dim \Lambda^n V = \binom{m}{n}$.

For any $g \in GL(V)$, the action of g on V extends linearly to $\Lambda(V)$, and so for any n we get a homomorphism

$$\rho_n : GL(V) \longrightarrow GL(\Lambda^n V).$$

- $\rho_n(g)(v_1 \wedge \cdots \wedge v_n) := g \cdot v_1 \wedge \cdots \wedge g \cdot v_n$
- Let $e_1, \dots, e_m$ be a basis of V, then $\{ e_I \mid \underset{|I|=n}{I \subseteq [1..m]} \}$ is a basis of $\Lambda^n V$ and

$$\rho(g)\, e_I = \sum_{\substack{J \subseteq [1..m] \\ |J| = n}} \det(g_{IJ})\, e_J$$

where $g_{IJ} = \begin{bmatrix} g_{i_1 j_1} & \cdots & g_{i_n j_1} \\ \vdots & & \vdots \\ g_{i_1 j_n} & \cdots & g_{i_n j_n} \end{bmatrix}$, $I = \{ i_1 < \cdots < i_n \}$, $J = \{ j_1 < \cdots < j_n \}$, and

$$g e_i = \sum_{j=1}^{m} g_{ij} e_j,$$

<u>Proposition.</u> Let W be a subspace of a finite-dimensional space V. Then

$$\{g\in GL(V) \mid gW=W\} = \{g\in GL(V) \mid \rho_d(g)(\wedge^d W)=\wedge^d W\}$$

where $d=\dim W$.

Pf. Let $e_1,\dots,e_d$ be a basis of W and $e_1,\dots,e_m$ be a basis of V. Then

$$\rho_d(g)(e_1\wedge\cdots\wedge e_d)=\sum_{\substack{J\subseteq[1..m]\\ |J|=d}} \det(g_{[1..d],J})\, e_J$$

- $gW=W \Rightarrow \rho_d(g)(\wedge^d W) = \wedge^d W$.
- Suppose $gW\neq W$ and $\rho_d(g)\, e_{[1..d]} = c_g\, e_{[1..d]}$.

Then the matrix of g in the basis $\{e_1,\dots,e_m\}$ is of the form

$$\begin{array}{cc} & \xleftrightarrow{\;d\;} \\ d\updownarrow & \left[\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right] \end{array}, \quad B\neq 0,$$

$\det A = c_g \neq 0$ and $\det X=0$ if $X\neq A$ and X is a $d\times d$ submatrix of $\left[\begin{array}{c} A \\ \hline B \end{array}\right]$. $\circledast$

Let v be a non-zero row of B. Then v is in the row span of $A=\begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}$ (A is invertible). So $\exists\, i$ s.t.

$\begin{bmatrix} v_1 \\ \hat{v_i} \\ \vdots \\ v_d \\ v \end{bmatrix}$ is invertible (why), which contradicts $\circledast$.

Proof of Proposition S

We have already found an algebraic group homomorphism $\rho: G \to GL(V)$ and a subspace $W \subseteq V$ s.t.

$$H = \{g \in G \mid \rho(g) W = W\}.$$

Now consider

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[rr, bend right, "\pi"'] & GL(V) \arrow[r, "\rho_d"] & GL(\Lambda^d V) \end{tikzcd}$$

and $\ell := \Lambda^d W \subseteq \Lambda^d V$.

Then $\{g \in G \mid \pi(g)(\ell) = \ell\} = \{g \in G \mid \rho_d(\rho(g))(\Lambda^d W) = \Lambda^d W\}$

$= \{g \in G \mid \rho(g)(W) = W\} = H.$

We also notice that $\rho_d: GL(V) \to GL(\Lambda^d V)$ is an algebraic group homomorphism (why?). So $\pi: G \to GL(\Lambda^d V)$ is an algebraic group homomorphism. ■

<u>Corollary</u>. Let G be an affine algebraic group and H be a closed subgroup of G. Then

$\exists$ a quasi-projective homogeneous space X for G together with a point $x_0 \in X$ s.t.

ⓐ $G_{x_0} := \{g \in G \mid g \cdot x_0 = x_0\} = H$

ⓑ The fibers of $g \overset{\psi}{\longmapsto} g \cdot x_0$ are cosets of H.

<u>Pf.</u> Let $\rho: G \to GL(V)$ and line $\ell \subseteq V$ be as in Prop. §

Then $G.[\ell] \subseteq \mathbb{P}(V)$ is open in $\overline{G.[\ell]}$. So $X := G.[\ell]$ is a quasi-projective variety. Let $x_0 := [\ell] \in X$. Then $G_{x_0} = H$ by Prop. S. Part (b) is a conseq. of part (a) and the fact that X is the G-orbit of x_0. ■

We would like to prove a universal property for the quotient space G/H:

<u>Theorem</u>. $\exists!$ a homogeneous space G/H for G together with a point x_0 s.t.:

If $G \curvearrowright Y$ and $\mathrm{Stab}(y_0) = H$ for some $y_0 \in Y$, then $\exists!$ G-equivariant morphism $G/H \xrightarrow{\Phi} Y$ s.t. $\Phi(x_0) = y_0$.

Notice that the above corollary gives us a homogeneous space X and a point x_0 s.t. for (Y, y_0) as in the above theorem we get a G-equivariant <u>map</u> $X \xrightarrow{\Phi} Y$ s.t. $\Phi(x_0) = y_0$. We have to show this <u>map</u> is actually a <u>morphism</u>.

For this we need more background from Algebraic Geometry.