Actions of algebraic groups

Monday, April 24, 2017 10:50 PM

Let $G \subseteq k^n$ be an affine algebraic group and X be a variety.

We say G acts on X, denote it by $G \curvearrowright X$,

if $\exists$ a morphism $G \times X \longrightarrow X$ s.t. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
$(g, x) \longmapsto g \cdot x$

What can we say about the orbits?

Lemma. (a) An orbit $G \cdot x$ is open in its closure.

(b) There exist closed orbits.

Pf. (a) $g \longmapsto g \cdot x_0$

$G \xrightarrow[\phi_{x_0}]{} X$ a morphism $\Big\} \Rightarrow$ by Chevalley's theorem, $\exists \ \emptyset \neq U \subseteq \operatorname{Im} \phi_{x_0}$ open in $\overline{\operatorname{Im} \phi_{x_0}}$.

$\Rightarrow \forall x \in G \cdot x_0, \ \exists \ U_x$ open $x \in U_x \subseteq G \cdot x_0$

$\overset{(?)}{\Rightarrow} \exists \ x_1, \ldots, x_m : \ G \cdot x_0 = \bigcup_{i=1}^{m} U_{x_i}$. $\Rightarrow G \cdot x_0$ is open.

(b) $\overline{G \cdot x_0} \setminus G \cdot x_0$ is closed and G-invariant.

$\Sigma := \{ Y \subseteq X \mid Y$ closed and G-invariant$\}$ has a minimal set Y_0.

Let $x_0 \in Y$. If $G \cdot x_0 \neq Y$, then $\underbrace{\overline{G \cdot x_0} \setminus G \cdot x_0}_{\text{in } \Sigma} \subsetneq Y$

which is a contradiction. ∎

Theorem. Let G be an affine algebraic group and X be an affine variety. Suppose $G \curvearrowright X$. Then

① $s : G \longrightarrow GL(k[X])$, $s(g)(f)(x) := f(g^{-1} \cdot x)$

is a group homomorphism; i.e. G acts linearly on $k[X]$.

② Let $m^* : k[X] \longrightarrow k[G] \otimes k[X]$ be the co-action map.

Then $W \subseteq k[X]$ is G-invariant $\Leftrightarrow m^*(W) \subseteq k[G] \otimes W$.

③ $\forall$ finite-dimensional subspace V of $k[X]$, there is a finite-dimensional subspace W of $k[X]$ s.t.

ⓐ $V \subseteq W$ ⓑ W is G-invariant.

Pf. ① Since $G \times X \xrightarrow{m} X$ and $G \xrightarrow{i} G$ are morphisms, $s(g)(f)$ is in $k[X]$. It is easy to check that $f \longmapsto s(g)(f)$ is an action, and $s(g)$ is linear.

② Let $\{f_i\}_{i \in I}$ be a k-basis of W, and let's extend it to $\{f_i\}_{i \in I \sqcup J}$ to get a k-basis of $k[X]$.

Let $m^*(f_i) = \sum_{j \in I \sqcup J} \xi_{ij} \otimes f_j$. Then for any $g \in G$ we have $m^*(f_i)(g, x) = \sum \xi_{ij}(g) f_j(x)$

$\Rightarrow f_i(g \cdot x) = \sum \xi_{ij}(g) f_j(x)$

$\Rightarrow s(g^{-1})(f_i) = \sum \xi_{ij}(g) f_j$.

$$\Rightarrow \quad s(g^{-1})(f_i) = \sum_{j \in I \sqcup J} \xi_{ij}(g)\, f_j .$$

$$\Rightarrow \quad s(g^{-1})(f_i) \in W \quad \text{if and only if} \quad \xi_{ij}(g) \quad \text{for } j \in J.$$

Hence $\begin{array}{l} g \cdot f_i \in W \\ \forall g \in G \end{array} \iff \xi_{ij} = 0 \quad \forall j \in J$

$\iff m^*(f_i) \in k[G] \otimes W.$

③ $\forall f \in k[X]$, $m^*(f) = \sum_{i=1}^{m} \xi_i \otimes f_i \Rightarrow$

$\forall g \in G$, $s(g^{-1})(f) = \sum \xi_i(g)\, f_i \in \text{Span}(f_1, \dots, f_m)$.

So the G-mod. generated by f is finite-dimensional. ■

<u>Theorem.</u> Any affine algebraic group is a linear algebraic group.

<u>Pf.</u> Let G be an affine algebraic group. Then $G \curvearrowright G$ by left multiplication. So by the above theorem, part (3), there are linearly independent functions $f_1, \dots, f_m \in k[G]$ s.t.

① $\text{Span}(f_1, \dots, f_m)$ is G-invariant.

② $k[G] = k[f_1, \dots, f_m]$.

So $\exists\ \xi_{ij} \in k[G]$ s.t. $s(g)(f_i) = \sum_{j=1}^{m} \xi_{ij}(g)\, f_j$.

- Let $\theta : G \longrightarrow M_m(k)$ be $\theta(g) = [\xi_{ij}(g)]$.

Since $s(g_1 g_2) = s(g_1) \circ s(g_2)$ and $s(g) = s(g)^{-1}$, we get that $\theta : G \to GL_m(k)$ is a group homomorphism.

- Since $\xi_{ij} \in k[G]$, θ is a morphism of algebraic sets. So θ is an affine algebraic group homomorphism. Hence $\operatorname{Im}(\theta) =: \overline{G}$ is an algebraic subgroup of $GL_m(k)$.

- $\theta^* : k[\overline{G}] \to k[G]$ is an isomorphism (?)

  - Since $\overline{G} = \theta(G)$, θ^* is injective.

  - $\theta^*(\underline{x_{ij}}) = \xi_{ij}$ and $f_i(g) = \sum_{j=1}^{m} \xi_{ij}(g^{-1}) f_j(e)$

    Let $y_{ij} = i^*(\underline{x_{ij}})$ where $i^* : k[GL_n] \to k[GL_m]$ is the co-inverse map. Then $\theta^*(y_{ij})(g) = \xi_{ij}(g^{-1})$.

    $\Rightarrow f_i \in$ Span of ξ_{ij}'s. $\Rightarrow \theta^*$ is surjective. ∎

<u>Remark</u>. As we mentioned earlier, it is NOT enough to check

$\ker(\theta) = I$:

$$SL_p(k) \to PGL_p(k) \quad \text{has trivial kernel and it is}$$
$$g \mapsto \operatorname{Ad}(g)$$

surjective if k is algebraically closed and $\operatorname{char}(k) = p$, but it is NOT an isomorphism.