LECTURE 5.

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Lemma 1. *Let I be an ideal of R. Consider the abelian additive group R/I. Then the following is a well-defined operation*
$$(a+I)\cdot(b+I) := (ab)+I.$$
Moreover $(R/I,+,\cdot)$ is a ring.

Proof. **Well-defined:** we have to show that if $a+I=a'+I$ and $b+I=b'+I$, then $(ab)+I=(a'b')+I$. It is equivalent to say that if $a-a'\in I$ and $b-b'\in I$, then $ab-a'b'\in I$:
$$ab-a'b'=(ab-ab')+(ab'-a'b')=a(b-b')+(a-a')b'\in RI+IR\subseteq I.$$
It is straightforward to check that it is a ring. □

Corollary 2. *Let $f:R\to R/I$, $f(a):=a+I$. Then f is a ring homomorphism and* $\ker(f)=I$.

Proof. It is a direct corollary of Lemma 1 that f is a ring homomorphism. We also have
$$\ker(f):=\{a\in R|\ f(a)=0\}=\{a\in R|\ a+I=0+I\}=I.$$
□

Corollary 3. *There is a correspondence between ideals and kernels of ring homomorphisms.*

Definition 4. A ring homomorphism $f:R\to S$ is called an isomorphism if it is a bijection.

Lemma 5. *If $f:R\to S$ is a ring isomorphism, then $f^{(-1)}:S\to R$ is also an isomorphism.*

Proof. Since $f^{(-1)}$ is clearly a bijection, it is enough to prove that it is a ring homomorphism:
$$f(f^{(-1)}(x)+f^{(-1)}(y))=f(f^{(-1)}(x))+f(f^{(-1)}(y))=x+y$$
and
$$f(f^{(-1)}(x)f^{(-1)}(y))=f(f^{(-1)}(x))f(f^{(-1)}(y))=xy.$$
Hence
$$f^{(-1)}(x)+f^{(-1)}(y)=f^{(-1)}(x+y)\text{ and }f^{(-1)}(x)f^{(-1)}(y)=f^{(-1)}(xy).$$
□

Lemma 6. *$f\in\hom(R,S)$ is an isomorphism if and only if* $\mathrm{Im}(f)=S$ *and* $\ker(f)=\{0\}$.

Proof. It is enough to show that a homomorphism is injective if and only if $\ker(f)=\{0\}$.

If $x\in\ker(f)$, then $f(x)=f(0)$. So if f is injective, then $x=0$.

If $f(x)=f(y)$, then $f(x-y)=0$,which means $x-y\in\ker(f)$. So if $\ker(f)=\{0\}$, then $x-y=0$, i.e. $x=y$. Hence f is injective. □

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