LECTURE 4.

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We started with recalling the definitions of (left) zero-divisor, integral domain, division ring and field.

Example 1.
(1) *If $a \in U(R)$, then a is not a zero-divisor.*
(2) *If R is a division ring, then it has no (left) zero-divisor. In particular, any field is an integral domain.*
(3) *$\mathbb{Z}$ is an integral domain which is not a field.*
(4) *$2\mathbb{Z}$ is NOT an integral domain (though it has no zero-divisor) (no unity!).*
(5) *$\mathbb{Z}/n\mathbb{Z}$ is an integral domain if and only if n is prime.*

Lemma 2. *Assume that R is a ring with no left zero-divisors. If $a \neq 0$ and $ax = ay$, then $x = y$.*

Proof.

$$\begin{aligned} ax = ay &\Rightarrow ax - ay = 0 \\ &\Rightarrow a(x-y) = 0 \\ &\Rightarrow x - y = 0 \text{ since } a \text{ is not a left zero-divisor.} \\ &\Rightarrow x = y \end{aligned}$$

□

Lemma 3. *If R is a finite integral domain, then it is a field.*

Proof. Let a be a non-zero element of R. Let $l_a : R \to R$, $l_a(x) := ax$. Then by Lemma 2 we have that l_a is injective (a.k.a. one-to-one). Since R is finite and l_a is injective, it is also surjective (a.k.a. onto). In particular, 1 is in the image of l_a, i.e. a is invertible. Hence $U(R) = R \setminus \{0\}$. On the other hand, R is commutative, which completes the proof. □

One of the important subrings of a unital ring is $S = \{n1_R | \, n \in \mathbb{Z}\}$. Let us define the characteristic of a ring and see its connection with this subring.

Definition 4. The smallest positive integer n is called the characteristic of a ring R if $nx = 0$ for any $x \in R$. If there is no such positive integer, we say that the characteristic of R is 0.

Lemma 5.
(1) *If $\mathrm{char}(R) = n \neq 0$, then $\mathrm{char}(R) = \mathrm{ord}(1)$ (here ord is the additive order.).*
(2) *If $\mathrm{ord}(1)$ is finite, then $\mathrm{char}(R) = \mathrm{ord}(1)$.*

Proof. 1. By the definition $n1 = 0$. Thus $\mathrm{ord}(1) \leq n$. On the other hand, for any $x \in R$ we have

$$\mathrm{ord}(1)x = (\mathrm{ord}(1)1) \cdot x = 0 \cdot x = 0. \qquad (1)$$

Therefore $\mathrm{ord}(1) \geq n$. Hence $\mathrm{ord}(1) = n$.

2. By the definition of characteristic, $\mathrm{char}(R) \geq \mathrm{ord}(1)$ and by Equation(1), we have $\mathrm{ord}(1) \geq \mathrm{char}(R)$. □

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